32nd International Conference
ENGINEERING MECHANICS 2026
Medlov, Czech Republic, May 11 – 13, 2026

POST-CRITICAL STABILITY OF STATIONARY SOLUTIONS IN A 2DOF GENERALIZED VAN DER POL AEROELASTIC MODEL

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Abstract: *This paper analyzes post-critical stationary solutions of a two-degree-of-freedom (2DOF) aeroelastic system modeled by a generalized van der Pol formulation. The sign of the linear damping term is shown to govern the qualitative character of instability, distinguishing cases with stable trivial equilibrium from self-excited oscillatory regimes corresponding to flutter- and galloping-type behavior. Stationary responses are derived using harmonic balance, yielding generalized modal amplitudes and multiplicative coupled solutions. The stability of trivial and nontrivial branches is examined analytically. Higher-order nonlinear damping terms of fourth and sixth degree are shown to ensure bounded post-critical motion and suppress excessive vibration growth. The linear stability boundary is related to Routh–Hurwitz conditions of the underlying 2DOF aeroelastic system, identifying flutter limits as the onset of Hopf bifurcation. The proposed framework provides a unified analytical interpretation of post-critical aeroelastic oscillations and their nonlinear saturation mechanisms.*

Keywords: Aeroelasticity, Flutter, Galloping, Van der Pol oscillator, Nonlinear damping

1. Introduction

Aeroelastic oscillations of bluff structures, such as bridge decks or slender beams in cross-flow, arise from the nonlinear interaction between structural motion and aerodynamic forces. Even within simplified two-degree-of-freedom (2DOF) models involving heave and pitch, the system exhibits a rich spectrum of stability phenomena, including flutter, divergence, and galloping-type instabilities (Strømmen, 2006).

Stability boundaries of coupled heave–pitch systems can be described using Routh–Hurwitz conditions in the frequency domain (Náprstek and Pospíšil, 2012). However, linear theory is insufficient for describing post-critical behavior, where nonlinear mechanisms determine whether oscillations grow unboundedly or settle into stable limit cycles. A convenient reduced-order representation of such is provided by generalized van der Pol–type equations. In these models, the linear damping term may change sign depending on flow velocity, while higher-order nonlinear terms provide amplitude limitation(Dowell, 2022).

In aeroelastic interpretation, regimes differing in the sign of the linear damping term correspond to physically distinct instability mechanisms. When the trivial solution loses stability due to coupled modal interaction and gyroscopic or nonconservative effects, the system undergoes flutter-type instability, characterized by oscillatory growth of coupled modes (Náprstek and Pospíšil, 2012), for nonlinear setting see (Náprstek and Fischer, 2020). In contrast, when negative aerodynamic damping dominates in a single mode, the response resembles galloping-type behavior, where nonlinear aerodynamic damping limits the oscillation amplitude to finite values (Vio et al., 2007).

Quadratic nonlinearities provide the primary saturation mechanism typical of supercritical flutter or galloping. However, if the linear damping term becomes negative—implying an unstable trivial equilibrium—additional higher-order damping terms are necessary to prevent unbounded amplitude growth and to establish a stable finite-amplitude limit cycle. Then the sixth-degree term acts as a global stabilizing contribution, ensuring bounded solutions even under strong negative linear damping (Dowell, 2022).Conversely, when the linear damping coefficient is positive, a fourth-degree nonlinear term may produce an unstable limit cycle surrounding the origin.

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ISSN: 1805-8256, doi: 10.21495/em2026-45

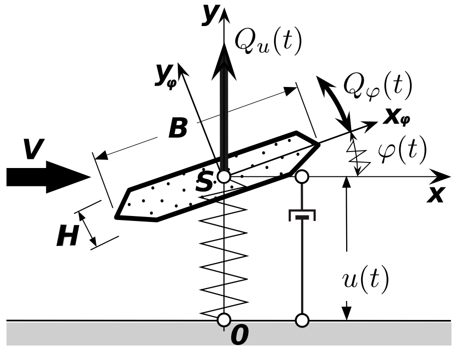

Fig. 1: Schematic representation of TDOF system and its aerodynamic excitation

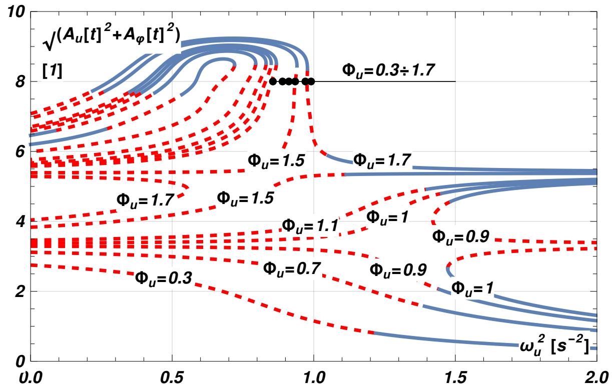

Fig. 2: Stable/unstable stationary generalised response amplitudes versus ω_u^2 for the positive linear damping case

2. Mathematical model

The governing equations of motion of a coupled aeroelastic system with two generalized coordinates corresponding to heave ($\bar{u}$, [m]) and pitch motion ($\bar{\varphi}$, [rad]) of a bluff cross-section are written as

$$m\,\ddot{\bar{u}} + c_u\big(1 - \nu_u u^2 + \vartheta_u u^4 - \zeta_u u^6\big)\,\dot{\bar{u}} + k_u \bar{u} - \bar{p}\,\bar{\varphi} - \bar{h}\,\bar{q}\,\dot{\bar{\varphi}} = \bar{F}_u(t), \tag{1}$$

$$I\,\ddot{\bar{\varphi}} + c_\varphi\big(1 - \nu_\varphi \varphi^2 + \vartheta_\varphi \varphi^4 - \zeta_\varphi \varphi^6\big)\,\dot{\bar{\varphi}} + k_\varphi \bar{\varphi} + \bar{g}\,\bar{p}\,\bar{u} + \bar{q}\,\dot{\bar{u}} = \bar{M}_\varphi(t), \tag{2}$$

where m [kg m^{-1}] and I [kg m] denote the generalized mass and generalized mass moment of inertia per unit span, respectively; c_u [kg (m s)$^{-1}$] and c_φ [kg s^{-1}] are damping coefficients; k_u [kg s^{-2}] and k_φ [kg m s^{-2}] are stiffness coefficients; $\bar{F}_u(t)$ [N m^{-1}] and $\bar{M}_\varphi(t)$ [N] are the generalized external force and moment, respectively. The coefficients $\bar{g}$ and $\bar{h}$ are dimensionless parameters controlling the relative contribution of the coupling terms, while $\bar{p}$ [kg s^{-2}] and $\bar{q}$ [kg s^{-1}] are stiffness-like and damping-like coupling coefficients, respectively. The nonlinear damping terms are introduced phenomenologically to represent combined structural and aerodynamic effects and depend on the dimensionless generalized coordinates

$$u(t) = \bar{u}(t)/H, \qquad \varphi(t) = \bar{\varphi}(t), \tag{3}$$

where H is the height of the cross-section, see Fig. 1. Since the radian is dimensionless $\varphi(t) = \bar{\varphi}(t)$.

The dimensionless formulation of the model is then obtained by substituting (3) into (1)–(2)

$$\begin{aligned} &\ddot{u} + b_u\left(1 - \nu_u u^2 + \vartheta_u u^4 - \zeta_u u^6\right)\dot{u} + \omega_u^2 u - p\,\varphi - hq\,\dot{\varphi} = Q_u(t),\\ &\ddot{\varphi} + b_\varphi\left(1 - \nu_\varphi \varphi^2 + \vartheta_\varphi \varphi^4 - \zeta_\varphi \varphi^6\right)\dot{\varphi} + \omega_\varphi^2 \varphi + gp\,u + q\,\dot{u} = Q_\varphi(t), \end{aligned} \tag{4}$$

where the normalized parameters are defined as

$$b_u = \frac{c_u}{m}, \quad \omega_u^2 = \frac{k_u}{m}, \quad b_\varphi = \frac{c_\varphi}{I}, \quad \omega_\varphi^2 = \frac{k_\varphi}{I}, \quad p = \frac{\bar{p}}{mH}, \quad gp = \frac{\bar{g}\,\bar{p}\,H}{I}, \quad q = \frac{\bar{q}\,H}{I}, \quad hq = \frac{\bar{h}\,\bar{q}}{mH},$$

and the excitation terms Q_u and Q_φ represent generalized accelerations:

$$Q_u(t) = \bar{F}_u(t)/(mH), \qquad Q_\varphi(t) = \bar{M}_\varphi(t)/I\,. \tag{5}$$

The external excitation is assumed harmonic

$$Q_u(t) = \Phi_u \cos(\Omega t), \qquad Q_\varphi(t) = \Phi_\varphi \cos(\Omega t)\,, \tag{6}$$

where Φ_u and Φ_φ are amplitudes of generalized acceleration and Ω is the excitation frequency.

The solution is sought in the form

$$u(t) = A_u(t)\cos\left(\Omega t + \psi_u(t)\right), \qquad \varphi(t) = A_\varphi(t)\cos\left(\Omega t + \psi_\varphi(t)\right), \tag{7}$$

where the dimensionless amplitudes A_u, A_φ and phases ψ_u, ψ_φ are assumed to vary on a slow time scale, i.e. their characteristic variation time is much larger than the oscillation period $2\pi/\Omega$.

To eliminate redundancy in the amplitude–phase representation, the orthogonality conditions are imposed:

$$\dot{A}_u \cos(\Omega t + \psi_u) - A_u \dot{\psi}_u \sin(\Omega t + \psi_u) = 0, \tag{8}$$

and analogously for the φ coordinate.

Substituting (7) into (4) and averaging over one excitation period $2\pi/\Omega$ yields a slowly-time system governing the evolution of amplitudes and phase differences ($\Delta = \psi_u - \psi_\varphi$):

$$\dot{\psi}_u = -\frac{(p\cos\Delta + hq\Omega\sin\Delta)}{2\Omega}\frac{A_\varphi}{A_u} - \frac{\left(\Omega^2 - \omega_u^2\right)}{2\Omega} - \frac{\Phi_u\cos\psi_u}{2\Omega A_u}\,, \tag{9a}$$

$$\dot{A}_u = \frac{(hq\Omega\cos\Delta - p\sin\Delta)\,A_\varphi}{2\Omega} - \frac{b_u A_u}{128}\left(64 - 16\nu_u A_u^2 + 8\vartheta_u A_u^4 - 5\zeta_u A_u^6\right) - \frac{\Phi_u\sin\psi_u}{2\Omega}\,, \tag{9b}$$

$$\dot{\psi}_\varphi = \frac{(gp\cos\Delta - q\Omega\sin\Delta)}{2\Omega}\frac{A_u}{A_\varphi} - \frac{15}{128A_\varphi}b_\varphi\zeta_\varphi A_u^6 A_\varphi\sin 2\Delta - \frac{\left(\Omega^2 - \omega_\varphi^2\right)}{2\Omega} - \frac{\Phi_\varphi\cos\psi_\varphi}{2\Omega A_\varphi}\,, \tag{9c}$$

$$\dot{A}_\varphi = -\frac{(q\Omega\cos\Delta + gp\sin\Delta)\,A_u}{2\Omega} - \frac{15}{128}b_\varphi\zeta_\varphi A_u^6 A_\varphi\cos 2\Delta - \frac{b_\varphi A_\varphi}{32}\left(16 - 4\nu_\varphi A_\varphi^2 + 2\vartheta_\varphi A_\varphi^4 - 5\zeta_\varphi A_u^6\right) - \frac{\Phi_\varphi\sin\psi_\varphi}{2\Omega}\,. \tag{9d}$$

For the purpose of stationary solution detection, the averaged slow-flow system is transformed into an autonomous algebraic form by introducing the substitutions $C_u = \cos\psi_u$, $S_u = \sin\psi_u$, $C_\varphi = \cos\psi_\varphi$, $S_\varphi = \sin\psi_\varphi$, together with the identities $C_u^2 + S_u^2 = 1, C_\varphi^2 + S_\varphi^2 = 1$. Setting the slow-flow derivatives equal to zero, $\dot{A}_u = 0, \dot{A}_\varphi = 0, \dot{\psi}_u = 0, \dot{\psi}_\varphi = 0$, leads to a closed algebraic system in the variables $A_u,\ A_\varphi,\ C_u,\ S_u,\ C_\varphi,\ S_\varphi$ and yields the stationary (periodic) solutions of the original nonlinear problem.

Local stability of a stationary solution is assessed using the original four-dimensional slow-flow system Eq. (9), namely the characteristic polynomial $\chi(\lambda)$ of the corresponding Jacobian has the form:

$$\chi(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4. \tag{10}$$

According to the Routh–Hurwitz criterion, the stationary solution is asymptotically stable if and only if

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_4 > 0, \qquad a_1a_2a_3 > a_3^2 + a_1^2a_4\,. \tag{11}$$

3. Example analysis

The stationary generalized amplitudes $(A_u^2 + A_\varphi^2)^{1/2}$ of the dimensionless response of the coupled 2DOF system obtained from Eq. (9) are illustrated in Figs. 2 and 3. The amplitudes are plotted as functions of the squared linear natural frequency ω_u^2, while the excitation frequency is kept fixed at $\Omega = 1$. The two cases differ in the character of linear damping.

Figure 2 corresponds to the case of positive linear damping $b_s = 0.2$, with parameter set $g = 1$, $h = 1$, $\nu_s = 0.5$, $\vartheta_s = 0.025$, $\Phi_\varphi = 0$, $p = 0.2$, $q = 0.2$ and biquadratic damping assumed ($\zeta_s = 0$); $s \in \{u, \varphi\}$ The remaining parameters are $\omega_\varphi^2 = 1$ and $\Omega = 1$.

Individual curves correspond to different excitation amplitudes Φ_u, indicated directly in the figure. For each value of Φ_u, the stationary amplitude varies nonlinearly with ω_u^2. Fold (turning) points appear on several branches, separating stable and unstable solution segments. Stable portions are plotted as solid blue curves, whereas unstable portions are shown as dashed red curves. In parameter intervals where folds occur, multiple stationary solutions coexist, indicating bistability and the possibility of jump phenomena under quasistatic parameter variation.

Increasing Φ_u raises the overall amplitude level and modifies the extent of stable and unstable regions. In this configuration, physically realizable stationary responses correspond to those branches satisfying the Routh–Hurwitz stability conditions.

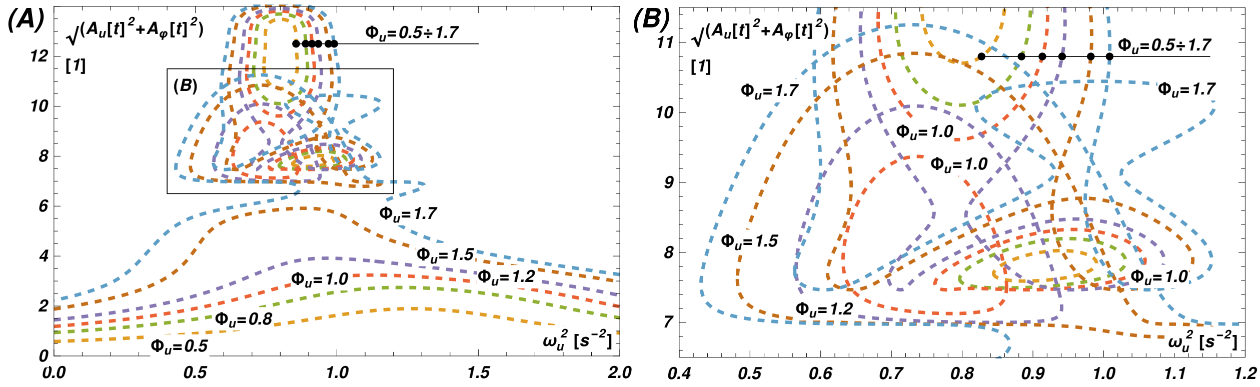

Fig. 3: (A) Stationary response amplitudes versus ω_u^2 for $\omega_\varphi^2 = 1$ and $\Omega = 1$ for negative linear damping $b_u, b_\varphi < 0$, shown for several excitation amplitudes Φ_u. (B) Detail indicated by a rectangle in plot (A).

Figure 3 shows the case of negative linear damping $b_u, b_\varphi = -0.2$. In contrast to the preceding case, the linear damping term supplies energy to the system and the trivial solution becomes unstable. To ensure bounded stationary amplitudes, a sixth–order nonlinear damping polynomial is introduced, $\vartheta_s = 0.00025$.

The figure presents potentially stationary amplitudes for $\omega_\varphi^2 = 0.85$, $\Omega = 1$, excitation amplitudes $\Phi_0 = 0.5,\ 0.8,\ 1.0,\ 1.2,\ 1.5,\ 1.7$ and $\omega_u^2 \in (0, 2)$. Plot (A) shows the global dependence of the stationary amplitudes, whereas plot (B) provides an enlarged detail of the region marked in plot (A), revealing the fine structure of closely spaced curves.

Although the sixth–order nonlinear damping term limits amplitude growth and produces finite solutions of the harmonic balance equations, all computed stationary branches in this parameter range are unstable according to the Routh–Hurwitz criteria. The plotted curves therefore represent mathematical stationary solutions of the algebraic system rather than physically stable steady states.

4. Conclusions

The nonlinear two-degree-of-freedom aeroelastic system was analyzed under harmonic external excitation for positive and negative values of linear damping. The model, formulated in a generalized van der Pol framework, was examined using harmonic balance and stability analysis.

When the excitation frequency is sufficiently far from the system eigenfrequencies, the response is predominantly forced and stationary, with self-excited effects remaining negligible. In contrast, when the excitation frequency approaches one or both eigenfrequencies, synchronization occurs. The system then exhibits large-amplitude stationary oscillations governed by frequency locking. In this regime, the synchronized response may become the only stable solution, representing a potentially dangerous operating condition.

The combined analytical and numerical approach proved effective in identifying synchronization domains and characterizing post-critical forced behavior. Future work will extend the analysis to coupled configurations and stochastic excitation.

Acknowledgments

The kind support of Czech Science Foundation project No. 24-13061S is gratefully acknowledged.

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