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GENERALIZED STANDARD MATERIALS FRAMEWORK AND ITS APPLICATION IN CONSTITUTIVE MODELING OF SHAPE MEMORY ALLOYS

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Abstract: *Advances in modern materials and manufacturing provide engineers with opportunities to pioneer innovative applications and redesign conventional concepts. Central to such an endeavor is a sound understanding of complex (thermo)mechanical behavior, which must be translated into reliable constitutive models for high-fidelity computational simulations. This work highlights the advantages of the Generalized Standard Materials framework—a systematic methodology that ensures a priori thermodynamic consistency and mathematical coherence, thereby offering a robust foundation for modeling advanced materials, where ad hoc approaches often prove inadequate.*

Keywords: Generalized standard materials, Internal variables, Dissipation potential, Shape memory alloys

1. Introduction

As materials science and technology provide engineers with novel materials — such as high-performance composites, additive-manufactured or functionally graded alloys, and architected metamaterials — the demand for sophisticated material modeling increases. By accurately capturing non-linearities, temporal dependencies, and multiphysical couplings, such modeling enables the design of optimized, resilient applications that push the boundaries of safety, efficiency, and innovation across a wide range of industries, from aerospace to biomedical.

Constitutive laws provide the backbone of efficient computational models. Within the general concept of continuum mechanics, the balance laws and side conditions (boundary, initial) are complemented by constitutive equations, which characterize the response of a material to imposed stimuli, so that the response of a solid body to external fields can be determined. The constitutive laws often account for inelastic deformation, which is related to microstructural changes in the material and is associated with energy dissipation, e.g., viscoelasticity, plasticity, damage, delamination, etc. Methods for developing constitutive laws in the form of mathematical equations range from purely phenomenological approaches based on simple observations to sophisticated frameworks anchored in thermodynamics. The latter approaches yield special classes of constitutive laws, see, e.g., (Truesdell and Noll, 1965; Coleman and Gurtin, 1967; Lubliner, 1969), and their advantage is that thermodynamical and mathematical properties of models in a particular class have already been investigated in a general manner, hence, they are known *a priori* for any newly developed member.

In this contribution, we briefly summarize the Generalized Standard Materials framework, introduced by Halphen and Nguyen (1975), which provides a rigorous basis for constructing a rich family of constitutive models. Standard in this context means that the entire thermomechanical behavior—both reversible and irreversible—can be derived from just two scalar potentials, and, under certain conditions, it is automatically consistent with the laws of thermodynamics. We also highlight the framework's benefits and provide illustrative examples with a particular focus on shape memory alloys.

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2. Thermodynamics preliminaries

In continuum theory, a material point is a small enough elementary particle of a solid or fluid, whose state represents the local state of the material. The local equilibrium thermodynamics then assumes that "the system is composed of infinitesimal subsystems in slow evolution such that each subsystem can always be considered as almost in thermodynamic equilibrium at any time" (Nguyen, 2000). Any material point is then understood to be composed of one of these infinitesimal subsystems, and the state of a material point in equilibrium can be characterized by the present value of a set of variables called state variables. The set of state variables includes the temperature, T, and other physical variables suitable for the investigated system.[2]

The mechanical response of an inelastically deforming material is linked to changes in its microstructure. Within the concept pioneered in (Coleman and Gurtin, 1967), the microstructure of the material (at the continuum level) can be fully described by a set of additional parameters called *internal state variables*, also termed just as internal variables, and "the present state of a material point depends only on the present values of macroscopic state variables and a set of internal variables" (Horstemeyer and Bammann, 2010).The macroscopic response of the material is then linked to the time evolution of the internal variables (Maugin and Muschik, 1994a).

Adopting the small strain setting for simplicity, we can choose temperature, T, and the small strain tensor, $\boldsymbol{\varepsilon}$, as state variables and denote the vector of all internal variables as $\boldsymbol{\alpha}$. Hence, internal energy, $u(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, entropy, $s(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, heat flux $\mathbf{q}(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, and the Cauchy stress, $\boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, are all considered functions of the state and internal variables.

Combining the local versions of the first and second laws of thermodynamics leads to the Clausius-Duhem inequality (being thus an equivalent form of the second law) in the form

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - (\dot{f} + s\dot{T}) - \mathbf{q} \cdot \frac{\nabla T}{T} \geq 0 \tag{1}$$

where the Helmholtz free energy, $f(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, is defined as a Legendre-Fenchel transformation of internal energy, $f = u - sT$, as a more convenient form for continuum mechanics of solids. The term

$$\mathcal{D}_{\text{mech}} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{f} - s\dot{T} \tag{2}$$

is called a *mechanical dissipation*, whereas the last term on the left-hand side of (1) is often called a *thermal dissipation* (Houlsby and Puzrin, 2000; Nguyen, 2000). It is usually reasonable to adopt the Fourier law for the heat conduction, i.e., to assume the heat flux density, $\mathbf{q}$, is proportional to a negative gradient of temperature, $-\nabla T$, with an appropriate proportionality coefficient so that the thermal dissipation is always non-negative. Requiring the mechanical dissipation itself to be non-negative

$$\mathcal{D}_{\text{mech}} \geq 0 \tag{3}$$

is then a slightly more stringent condition than the Clausius-Duhem inequality itself. However, it is a widely accepted assumption that provides sufficient space for advanced constitutive modeling.

Expanding (2) and inserting into (3) leads to

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial f}{\partial T}\dot{T} - \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} - s\dot{T} \geq 0 \tag{4}$$

Applying a strategy known as the Coleman-Noll procedure (Coleman and Gurtin, 1967), one gets the system:

$$\boldsymbol{\sigma} - \frac{\partial f}{\partial \boldsymbol{\varepsilon}} = \mathbf{0}, \tag{5}$$

$$s + \frac{\partial f}{\partial T} = 0, \tag{6}$$

$$-\frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} \geq 0. \tag{7}$$

[2] The above postulate allows us to introduce the entropy and thermodynamic potentials locally as functions of state variables, even when the system as a whole is not necessarily in equilibrium.

Equation (3) may be understood as a restriction imposed by the second law of thermodynamics on the time evolution of state variables, including the formally introduced additional internal variables. System (5)–(7) may then be viewed as a set of sufficient conditions for satisfying that restriction, useful for the construction of constitutive models.

Defining a vector of *generalized forces*

$$\boldsymbol{A} := -\frac{\partial f}{\partial \boldsymbol{\alpha}}, \tag{8}$$

the entropy production inequality (7) takes the form

$$\boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} \geq 0. \tag{9}$$

Let us note that $\dot{\boldsymbol{\alpha}}$ is termed as *generalized flux* or *generalized velocity* and A serves as a "driving force" for microstructural processes captured by $\dot{\boldsymbol{\alpha}}$ in this context.

The formulation of the constitutive equations now boils down to establishing the relation between A and $\dot{\boldsymbol{\alpha}}$. In addition to the ad hoc approach, systematic methods have been reported in the literature. They are often based on an extremum principle applied to a physical quantity related to entropy production (Fischer et al., 2014). Among the most well-known are the maximum entropy production principle (also known as the principle of maximum dissipation rate), whose roots can be traced back to Onsager, the minimum principle for the dissipation potential, the modern GENERIC framework, and others. In what follows, we focus on the approach widely used in the engineering mechanics of solids.

3. Generalized Standard Materials framework

Within the Generalized Standard Materials (GSM) methodology, the relation between A and $\dot{\boldsymbol{\alpha}}$ is resolved via defining an additional constitutive function, d, called *dissipation potential*[3] and the "normality rule", which states that the generalized force $\boldsymbol{A}$ belongs to the *subdifferential*[4] of the dissipation potential:

$$\boldsymbol{A} \in \partial d_{\dot{\boldsymbol{\alpha}}}(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}), \tag{10}$$

where $\boldsymbol{\chi}$ represents a set of state variables, i.e. $\{\boldsymbol{\varepsilon}, T\}$ in the situation at hand. In the differentiable case, this simplifies to

$$\boldsymbol{A} = \frac{\partial d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})}{\partial \dot{\boldsymbol{\alpha}}}. \tag{11}$$

The function $d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$ hence describes the kinetics of the irreversible processes. To fullfill (9), it is assumed to satisfy:

(A1) **Convexity:** $d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$ is convex in the rate(s) $\dot{\boldsymbol{\alpha}}$,

(A2) **Non-negativity:** $d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \geq 0$,

(A3) **Vanishing for zero rate(s):** $d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \boldsymbol{0}) = 0$.

Indeed, then clearly

$$\boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} \overset{(\mathrm{A1})}{\geq} d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) - d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \boldsymbol{0}) \overset{(\mathrm{A3})}{=} d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \overset{(\mathrm{A2})}{\geq} 0. \tag{12}$$

Thus, the Clausius-Duhem inequality is satisfied *by construction*, namely by virtue of satisfying (5)–(7).

The GSM [5] framework developed originally by Halphen and Nguyen (1975) hence allows for the construction of a rich family of constitutive models satisfying the Clausius-Duhem inequality. The key points for

[3] Alternative terms *dissipation functional*, *dissipation (rate) function*, *generalized stress potential*, *force potential*, etc. can be found depending on the author and context.

[4] Loosely speaking, subdifferential is a generalization of the gradient for non-smooth functions, i.e. the set of all possible slopes of tangent planes at a point on a convex function; see (Rockafellar, 1970) for details.

[5] Alternative terminology can also be found in the literature, e.g. *generalized standard models* (Germain et al., 1983) or *standard media* (Junker et al., 2014), *hyperplasticity* (Houlsby and Puzrin, 2000), etc.

establishing a model within GSM are: i) parametrization of dissipative processes by internal variables, $\boldsymbol{\alpha}$, and ii) specification of two scalar constitutive functions: one for energy storage, $f(\boldsymbol{\chi}, \boldsymbol{\alpha})$, and the other for energy dissipation, $d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$, which must satisfy the properties (A1)–(A3). Both functions may depend on state and internal variables. As pointed out by Nguyen (2000), a good understanding of the microstructural mechanisms operating within the considered material is essential for constructing a physically relevant model.

4. Benefits of GSM framework

Models constructed within the GSM framework constitute only a subclass of all possible material models. However, the framework appears to be highly useful and powerful, at least from the following perspectives.

4.1. Thermodynamics

As shown above, the requirements on the dissipation function (A1)–(A3) automatically ensure the consistency of any GSM model with the fundamental laws of thermodynamics. In fact, this general consistency was the natural motivation for the development of the concept. Moreover, in many common situations,[6] the relation (10) may be derived from so-called thermodynamic extremal principles, e.g., the minimum principle for the dissipation potential

$$\min_{\dot{\alpha}} \{\dot{f} + d\}, \tag{13}$$

with the dot denoting the time derivative, see (Hackl and Fischer, 2008; Fischer et al., 2014) for details.

Example. The dissipation potential of a rate independent process (e.g. elasto-plasticity) is a positive one-homogenous function[7], its dual has a specific form, cf. (Houlsby and Puzrin, 2000), and it holds

$$\boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} = d(\boldsymbol{\chi}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}). \tag{14}$$

Hence, equation (2) implies

$$d = \mathcal{D}_{\mathrm{mech}}, \tag{15}$$

i.e., the value of the prescribed potential, d, is equal to the mechanical part of the dissipation. This provides a clue for relating the hysteresis in the response to the dissipation potential.

4.2. Experimental physics

Internal state variables are often introduced with a specific motivation, i.e., they are related to some real parameters of the microstructure. Recent advances in materials characterization techniques extend the capabilities of direct imaging; hence, these techniques enable validation of model predictions not only in terms of macroscopic response but also of microstructural information.

Example. In (Frost et al., 2018, 2020), the computed evolution of the internal variable representing volume fractions of phases in a GSM model was directly compared with their experimental counterparts obtained with advanced in-situ three-dimensional X-ray tomographic techniques.

4.3. Applied mathematics and computations

The GSM framework unifies diverse physical behaviors under a single mathematical "roof". Thanks to the aforementioned relation to extremal principles, the mathematical structure of the evolutionary system is well-defined and allows for rigorous analysis within a suitable solution concept, e.g., (Mielke and Roubíček, 2015) for non-smooth d. The analysis can also provide a hint for a tailored numerical discretization scheme.

Example. Since the GSM framework complies well with variational formulations, the developed models can be implemented using optimization techniques. The starting point is the incremental energy functional J for a time step Δt evaluated at time increment $n+1$, i.e.

$$J(\Delta\boldsymbol{\alpha}) = f(\boldsymbol{\chi}_{n+1}, \boldsymbol{\alpha}_n + \Delta\boldsymbol{\alpha}) - f(\boldsymbol{\chi}_n, \boldsymbol{\alpha}_n) + d\left(\boldsymbol{\chi}_{n+1}, \boldsymbol{\alpha}_n, \frac{\Delta\boldsymbol{\alpha}}{\Delta t}\right)\Delta t. \tag{16}$$

[6] For instance, if f is convex and d is homogeneous (see also the next footnote).

[7] A function $g(x)$ is called *positive k-homogenous* with k being integer, if for any real $\lambda > 0$ and any x it holds $g(\lambda x) = \lambda^k g(x)$.

Tab. 1: GSM formulations for some simplified mechanical behaviors of solids in a one-dimensional material point. $\mathcal{I}_{\mathbb{R}_0^+}(\cdot)$ denotes the indicator function of non-negative reals (in the sense of convex analysis). Characters other than ε, T, and those marked as internal variables denote material parameters.

Material Model	**Int. Var. $\boldsymbol{\alpha}$**	**Free Energy f**	**Dissipation Potential d**
Linear Elasticity	–	$\frac{1}{2}E\varepsilon^2$	–
Perfect Elastoplasticity	$\varepsilon_{\mathrm{pl}}$	$\frac{1}{2}E(\varepsilon-\varepsilon_{\mathrm{pl}})^2$	$Y_{\mathrm{p}}\lvert\dot{\varepsilon}_{\mathrm{pl}}\rvert$
EP with Kinematic Hardening	$\varepsilon_{\mathrm{pl}}$	$\frac{1}{2}E(\varepsilon-\varepsilon_{\mathrm{pl}})^2+\frac{1}{2}\kappa\varepsilon_{\mathrm{pl}}^2$	$Y_{\mathrm{p}}\lvert\dot{\varepsilon}_{\mathrm{pl}}\rvert$
EP with Isotropic Hardening	$\varepsilon_{\mathrm{pl}}$	$\frac{1}{2}E(\varepsilon-\varepsilon_{\mathrm{pl}})^2$	$(Y_{\mathrm{p}}+\zeta\varepsilon_{\mathrm{pl}})\lvert\dot{\varepsilon}_{\mathrm{pl}}\rvert$
Brittle Damage	$\omega\in[0,1]$	$\frac{1}{2}E(1-\omega)\varepsilon^2$	$(Y_{\mathrm{d}}+\chi\omega)\dot{\omega}+\mathcal{I}_{\mathbb{R}_0^+}(\dot{\omega})$
Perfect Superelasticity	$\xi\in[0,1]$	$\frac{1}{2}E(\varepsilon-\xi\epsilon)^2+\upsilon(T-T_0)$	$Y_{\mathrm{pt}}\lvert\dot{\xi}\rvert$
Kelvin-Voigt Viscoelasticity	$\varepsilon_{\mathrm{vis}}$	$\frac{1}{2}E(\varepsilon-\varepsilon_{\mathrm{vis}})^2$	$\frac{1}{2}\eta\dot{\varepsilon}_{\mathrm{vis}}^2$

The state update is then obtained as $\boldsymbol{\alpha}_{n+1}-\boldsymbol{\alpha}_n=\Delta\boldsymbol{\alpha}=\arg\min J$, thereby transforming the relation (10) to a minimization problem (16). Advantageously, this can be further combined with the principle of minimum potential energy, yielding a compact, discrete variational formulation of the evolutionary boundary value problem (Hackl and Fischer, 2008) that is solvable, e.g., using the finite element method with a suitable numerical strategy, cf. (Frost and Valdman, 2022).

4.4. Continuum mechanics

Many common material models can be reformulated to fit the GSM framework, see Tab. 1 and (Maugin and Muschik, 1994b; Houlsby and Puzrin, 2000, 2002) for instance. More importantly, the variational approach of GSM provides a *systematic way* for the derivation of evolution equations for internal variables and, thus, "can also be applied to rather complex systems, for which the classical approach involving application of phenomenological equations fails" (Fischer et al., 2014). This is particularly the case of multiple, strongly coupled dissipation processes (Einav et al., 2007).

The mathematical theory of convex functions provides useful relations between formulations in terms of generalized forces and fluxes, which are beneficial for recasting the physical understanding of the system into concrete constitutive equations. Under relatively mild mathematical conditions[8] on d, relation (10) is equivalent to the *dual formulation*

$$\dot{\boldsymbol{\alpha}}\in\partial_{\boldsymbol{A}}d^*(\boldsymbol{\chi},\boldsymbol{\alpha},\boldsymbol{A}), \tag{17}$$

where the *dual dissipation potential*, d^*, is given by the Legendre-Fenchel transform of d as follows

$$d^*(\boldsymbol{\chi},\boldsymbol{\alpha},\boldsymbol{A}):=\sup_{\dot{\boldsymbol{\alpha}}}\{\boldsymbol{A}\cdot\dot{\boldsymbol{\alpha}}-d(\boldsymbol{\chi},\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}})\}. \tag{18}$$

Moreover,

$$d(\boldsymbol{\chi},\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}})+d^*(\boldsymbol{\chi},\boldsymbol{\alpha},\boldsymbol{A})=\boldsymbol{A}\cdot\dot{\boldsymbol{\alpha}}. \tag{19}$$

More details can be found in (Rockafellar, 1970).

Example. To illustrate the above relations and the way they can be used in developing constitutive laws, let us find a dissipation potential for a microstructurally transforming material inspired by (Richards et al., 2013). Let α be a scalar representing the volume fraction of a new constituent and A be the thermodynamic generalized force controlling the process. For material parameters $A_0\geq 0, c>0$, the kinetics of the process is prescribed by the following relation

$$\dot{\alpha}=\begin{cases} c(A-A_0) & \text{if} \quad A\geq A_0,\\ 0 & \text{if } -A_0<A<A_0,\\ c(A+A_0) & \text{if} \quad A\leq -A_0. \end{cases} \tag{20}$$

[8] For functions satisfying (A1)-(A3), it is additionally the weak lower semi-continuity.

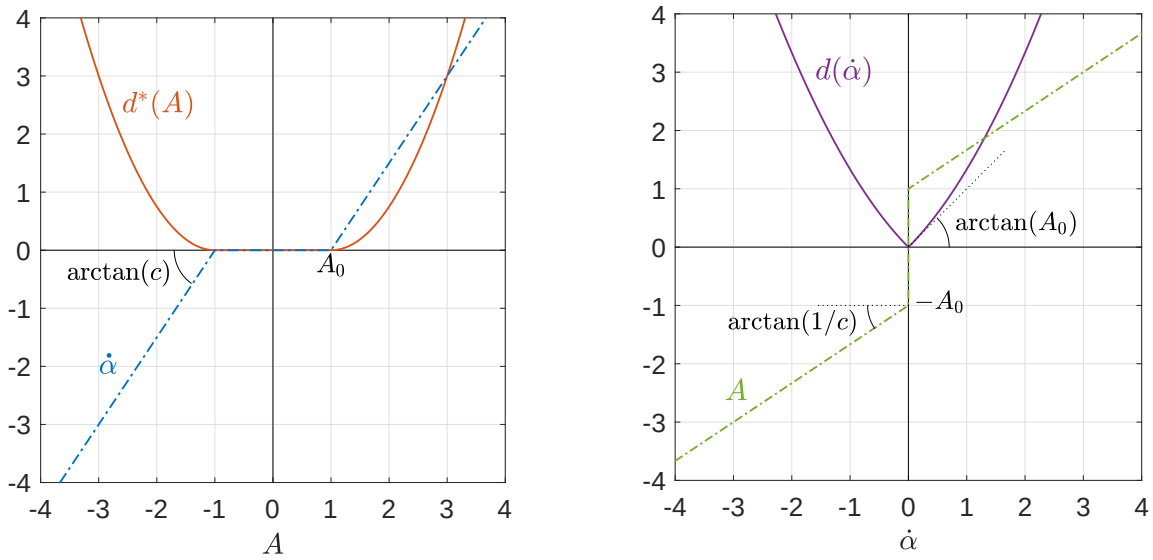

Fig. 1: Visualization of the kinetic relations and dissipation potentials treated in the example in subsection 4.4. with material parameters fixed to $A_0 = 1$ and $c = 1.5$.

Clearly, $A\dot\alpha \geq 0$ as required by (9). First, the dual form (i.e., the dissipation potential in terms of the generalized force) is for the fixed state variables resolved by applying (17) and employing (19), so one gets

$$d^*(A) = \begin{cases} \frac{c}{2}(A - A_0)^2 & \text{if} \quad A \geq A_0, \\ 0 & \text{if} \; -A_0 < A < A_0, \\ \frac{c}{2}(A + A_0)^2 & \text{if} \quad A \leq -A_0. \end{cases} \tag{21}$$

Then, (18) allows to recover the dissipation potential as

$$d(\dot\alpha) = A_0|\dot\alpha| + \frac{1}{2c}\dot\alpha^2. \tag{22}$$

Finally, the force-flux relation (10) takes the form

$$\begin{array}{ll} A = A_0 \frac{\dot\alpha}{|\dot\alpha|} + \frac{1}{c}\dot\alpha & \text{if} \;\; \dot\alpha \neq 0, \\ A \in [-A_0, A_0] & \text{if} \;\; \dot\alpha = 0. \end{array} \tag{23}$$

Note that a closed interval of values is admissible for the zero rate, see the second line. Such a response corresponds to an activated system with rate-dependent kinetics (i.e., of a visco-plastic type): the constant A_0 determines the critical force for activation of the process, and $1/c$ is the kinetics rate coefficient. Fig. 1 illustrates the derived relations. Let us note that for $A_0 = 0$, the process occurs even for an infinitesimal value of the driving force (i.e., becoming of a purely viscous type), which results in a completely smooth $d(\dot{\boldsymbol{\alpha}})$.

5. Modeling the thermomechanical response of shape memory alloys using GSM

Shape memory alloys (SMAs) are metallic materials that undergo a reversible solid-to-solid phase transformation between a high-temperature phase and a low-temperature phase. In addition to temperature changes, the transformation may be induced by applying stress to the high-temperature phase. The most common and widely used examples are alloys with nickel and titanium as the dominant constituents, known as NiTi-based SMAs, in which the two dominant phases are termed austenite and martensite, respectively.

From a mechanical standpoint, NiTi polycrystals exhibit exceptional features (Otsuka and Wayman, 1998). They can be strained up to several percent deformation and, upon unloading, fully recover their original shape; this is termed *superelasticity* and typically occurs at higher temperatures, where austenite is stable. If the material is deformed at low temperatures and its shape change is constrained, it generates a high recovery force against that constraint. Finally, loaded SMA structures can perform mechanical work when

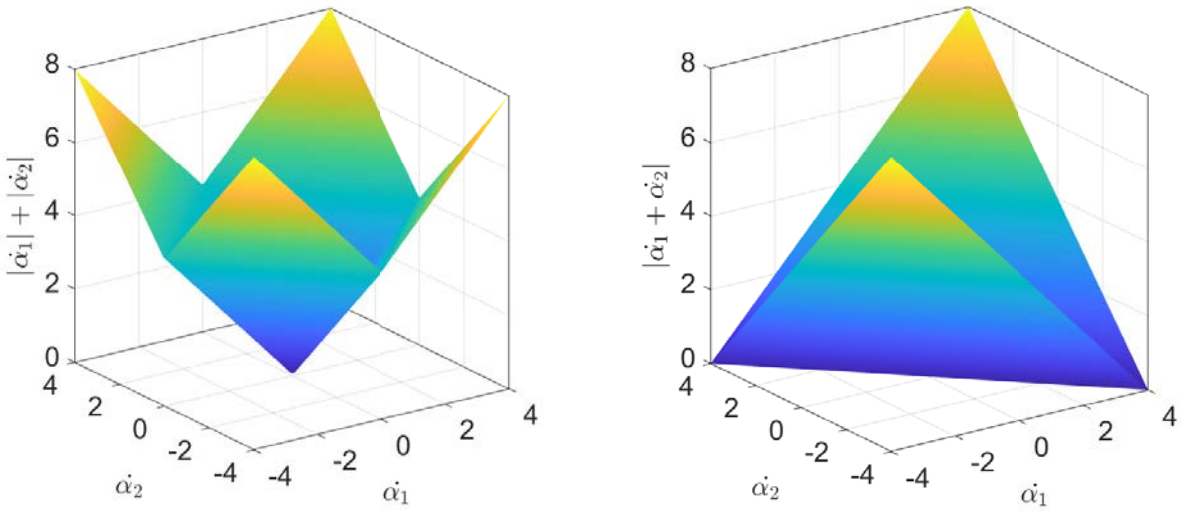

Fig. 2: The difference between the decoupled (left) and coupled (right) functions of the type used in the dissipation potential for SMAs is illustrated for a simpler situation with two scalar internal variables α_1, α_2 and functions $|\dot{\alpha}_1| + |\dot{\alpha}_2|$ and $|\dot{\alpha}_1 + \dot{\alpha}_2|$.

cycled between high and low temperatures. All these features have led to numerous smart and useful applications (Mohd Jani et al., 2014).

The mechanical response of NiTi exhibits many peculiarities, e.g., the pronounced asymmetry of the response in tension and compression, phase-dependency and anisotropy of properties, or a tendency towards strain localization. However, the fundamental challenge in constructing a robust and comprehensive constitutive model lies in the coupling among multiple dissipative processes associated with microstructural changes (phase transformation, (de)twinning, plasticity).

Based on the analysis of the hysteretic behavior and inspired by previous experimental findings, a particular form of the dissipation potential for the *fully reversible* behavior of NiTi SMAs was suggested in (Sedlák et al., 2012; Frost et al., 2016).[9] The vector of internal variables consists of two members: one scalar, ξ capturing the phase transformation, and the other tensorial, $\boldsymbol{\varepsilon}^{\mathbf{tr}}$, describing the martensitic microstructure reconfiguration (detwinning).[10] The dissipation potential takes the form

$$d_{\mathrm{SMA}}(T, \xi, \boldsymbol{\varepsilon}^{\mathrm{tr}}, \dot{\xi}, \dot{\boldsymbol{\varepsilon}}^{\mathrm{tr}}) = \begin{cases} c^{\mathrm{f}}(\xi)|\dot{\xi}| + k^{\mathrm{f}}(T)\|\dot{\xi}\boldsymbol{\varepsilon}^{\mathrm{tr}} + \xi\dot{\boldsymbol{\varepsilon}}^{\mathrm{tr}}\|, & \text{if } \dot{\xi} \geq 0, \quad (24) \\ c^{\mathrm{r}}(\xi)|\dot{\xi}| + k^{\mathrm{r}}(T)\left(\|\dot{\xi}\boldsymbol{\varepsilon}^{\mathrm{tr}}\| + \|\xi\dot{\boldsymbol{\varepsilon}}^{\mathrm{tr}}\|\right) & \text{if } \dot{\xi} < 0. \quad (25) \end{cases}$$

where $\|\cdot\|$ is (Frobenius) tensor norm. Material functions $c^{\mathrm{f}}(\xi), c^{\mathrm{r}}(\xi)$ and $k^{\mathrm{f}}(T), k^{\mathrm{r}}(T)$ are linear in their variables and all are chosen to ensure (A1)-(A3), cf. (Frost et al., 2025).

It is worth noting that the function d_{SMA} is 1-homogeneous in rates of internal variables, which corresponds to the rate-independent nature of both processes. The function depends on the "direction" of one process – the sign of $\dot{\xi}$ distinguishes the forward transformation (24) from the reverse one (25), and it is clearly *asymmetric* with respect to this. Moreover, due to the coupling of internal variables in the second term in (24), it is not possible to split the dissipation potential into two additive terms corresponding to the time evolution of the respective variables. This is in sharp contrast to the separation of processes that is common in uncoupled or weakly-coupled models, cf. (Einav et al., 2007). On the other hand, the separation of processes as in (25) allows for capturing the so-called mechanical stabilization effect in SMAs (Frost and Valdman, 2022).

The difference between the terms that combine both processes in d_{SMA} is illustrated in a much simpler setting in Fig. 2. Whereas the decoupled formula (left) is non-smooth only in situations where at least one flux vanishes, the coupled formula (right) is non-smooth even if both processes are active. Tackling non-

[9] An extension towards incorporating also the irreversible processes, namely plasticity, was suggested in (Sedlák et al., 2022).

[10] Let us also note that $\xi \in [0, 1]$ and $\boldsymbol{\varepsilon}^{\mathbf{tr}}$ is bounded to a convex set.

smoothness is crucial for the numerical implementation of such models, especially in methods that require gradients of the dissipation potential.

6. Conclusions

The GSM framework is a powerful tool for developing constitutive equations for materials behavior, particularly those with multiple interacting dissipative processes. The resulting models are thermodynamically consistent and mathematically rigorous; hence, they can be theoretically investigated using a priori developed methodologies and implemented in computational and simulation tools, thereby enabling sophisticated materials engineering and product design.

Nevertheless, as a "mere" tool, the GSM framework encourages researchers to draw on their creativity and materials-science insight to identify appropriate descriptors for the investigated material phenomena and to define the specific forms of the two constitutive functions. Consequently, this stimulates further enhancements, e.g., incorporation of (non-local) gradient theories (Nguyen, 2021) or analysis of GSM systems with two competing dissipative mechanisms (Mielke et al., 2025).

Acknowledgments

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References

Coleman, B. D. and Gurtin, M. E. (1967) Thermodynamics with internal state variables. *J. Chem. Phys.*, 47, pp. 597–613.

Einav, I., Houlsby, G. T., and Nguyen, G. D. (2007) Coupled damage and plasticity models derived from energy and dissipation potentials. *Int. J. Solids Struct.*, 44, pp. 2487–2508.

Fischer, F. D., Svoboda, J., and Petryk, H. (2014) Thermodynamic extremal principles for irreversible processes in materials science. *Acta Mater.*, 67, pp. 1–20.

Frost, M., Benešová, B., and Sedlák, P. (2016) A microscopically motivated constitutive model for shape memory alloys: formulation, analysis and computations. *Math. Mech. Solids*, 21, 3, pp. 358–382.

Frost, M., Moskovka, A., and Knapek, M. (2025) Modeling the hysteresis of the thermomechanical response of superelastic β-titanium alloys. In Adámek, V., Jonášová, A., and Plánička, S., eds, *Proceedings of Computational Mechanics 2025*. West Bohemian University, Plzeň, pp. 35–38.

Frost, M., Sedlák, P., Heller, L., Kadeřávek, L., and Šittner, P. (2018) Experimental and computational study on phase transformations in superelastic NiTi snake-like spring. *Smart Mater. Struct.*, 27, pp. 095005.

Frost, M. and Valdman, J. (2022) Vectorized MATLAB implementation of the incremental minimization principle for rate-independent dissipative solids using FEM: A constitutive model of shape memory alloys. *Mathematics*, 10, 23, pp. 4412.

Frost, M., Ševčík, M., Kadeřávek, L., Šittner, P., and Sedlák, P. (2020) Reconstruction of phase distributions in NiTi helical spring: comparison of diffraction/scattering computed tomography and computational modeling. *Smart Mater. Struct.*, 29, pp. 075036.

Germain, P., Nguyen, Q. S., and Suquet, P. (1983) Continuum thermodynamics. *J Appl Mechanics*, 50, pp. 1010–1020.

Hackl, K. and Fischer, F. D. (2008) On the relation between the principle of maximum dissipation and inelastic evolution given by dissipation potentials. *Proc. R. Soc. London, Ser. A*, 464, pp. 117–132.

Halphen, B. and Nguyen, Q. S. (1975) Sur les matériaux standard généralisés. *J. Mecanique*, 14, pp. 39–63.

Horstemeyer, M. F. and Bammann, D. J. (2010) Historical review of internal state variable theory for inelasticity. *Int. J. Plasticity*, 26, pp. 1310–1334.

Houlsby, G. T. and Puzrin, A. M. (2000) A thermomechanical framework for constitutive models for rate-independent dissipative materials. *Int. J. Plasticity*, 16, pp. 1017–47.

Houlsby, G. T. and Puzrin, A. M. (2002) Rate-dependent plasticity models derived from potential functions. *J. Rheol.*, 46, pp. 113–126.

Junker, P., Makowski, J., and Hackl, K. (2014) The principle of the minimum of the dissipation potential for non-isothermal processes. *Continuum Mech. Thermodyn.*, 26, pp. 259–268.

Lubliner, J. (1969) On fading memory in materials of evolutionary type. *Acta Mechanica*, 8, pp. 75–81.

Maugin, G. A. and Muschik, W. (1994a) Thermodynamics with internal variables. Part I. General concepts. *J. Non-Equilib. Thermodyn.*, 19, pp. 217–249.

Maugin, G. A. and Muschik, W. (1994b) Thermodynamics with internal variables. Part II. Applications. *J. Non-Equilib. Thermodyn.*, 19, pp. 250–289.

Mielke, A., Rossi, R., and Stephan, A. (2025) On time-splitting methods for gradient flows with two dissipation mechanisms. *Calc. Var.*, 64, pp. 63.

Mielke, A. and Roubíček, T. (2015) *Rate-Independent Systems: Theory and Application*. Applied Mathematical Sciences. Springer New York.

Mohd Jani, J., Leary, M., Subic, A., and Gibson, M. A. (2014) A review of shape memory alloy research, applications and opportunities. *Mater. Design*, 56, pp. 1078–1113.

Nguyen, Q. S. (2000) *Stability and Nonlinear Solid Mechanics*. John Wiley & Sons, New York.

Nguyen, Q. S. (2021) On standard gradient plasticity and visco-plasticity. *Int. J. Solids Struct.*, 225, pp. 111038.

Otsuka, K. and Wayman, C. M. (1998) *Shape Memory Materials*. Cambridge University Press.

Richards, A. W., Lebensohn, R. A., and Bhattacharya, K. (2013) Interplay of martensitic phase transformation and plastic slip in polycrystals. *Acta Mater.*, 61, 12, pp. 4384–4397.

Rockafellar, R. T. (1970) *Convex analysis*. Princeton University Press.

Sedlák, P., Frost, M., Benešová, B., Šittner, P., and Ben Zineb, T. (2012) Thermomechanical model for NiTi-based shape memory alloys including R-phase and material anisotropy under multi-axial loadings. *Int. J. Plasticity*, 39, pp. 132–151.

Sedlák, P., Frost, M., Seiner, H., Heller, L., and Šittner, P. (2022) Thermodynamical model of NiTi shape memory alloys including plastic deformation mechanisms. In *SMST 2022: Extended Abstracts from the International Conference on Shape Memory and Superelastic Technologies*, Carlsbad: USA. ASM International, pp. 65–66.

Truesdell, C. and Noll, W. (1965) *The Non-Linear Field Theories of Mechanics*. Springer, Berlin.