31st International Conference
ENGINEERING MECHANICS 2025
Medlov, Czech Republic, May 12 –14, 2025
Paper **#52,** pp. 77–80, doi: 10.21495/em2025-77

NUMERICAL APPROACH TO DESCRIBE FRACTIONAL VISCOELASTICITY

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Abstract: *Fractional viscoelasticity introduces a new approach to describe viscoelastic materials using derivatives and integrals of noninteger order. The main disadvantage of this approach is its use in numerical applications because, compared to standard integer-order derivatives, the numerical approximation of the fractional derivative is based on all of the nodal values from the previous time steps while each of these values has a different weight in the computation. For this reason, the usability for large-scale problems is limited because of high memory and computing time requirements. This paper presents several approaches how to deal with this problem by introducing approximation of the fractional derivative which uses only reduced number of previous values.*

Keywords: Springpot, Fractional viscoelasticity, Viscoelastic response, Numerical approximation of fractional derivative

1. Introduction

A common way to describe viscoelastic materials is through theoretical models consisting of elastic springs and viscous dashpots. Another possible approach is to use fractional viscoelasticity, which is based on fractional calculus, a theory of integrals and derivatives of noninteger order. An insight into this mathematical apparatus can be found, for example, in Oldham and Spanier (1974) or Becker and Purnaras (2018).

The fractional viscoelasticity introduces another rheological element, the springpot. Its behavior can be described by the following constitutive law

$$\sigma(t) = E\tau_{\mathrm{c}}^{\alpha} D^{\alpha}\varepsilon(t), \tag{1}$$

where σ denotes normal stress, ε normal strain and τ_{c} is a characteristic time, the ratio between viscosity η and Young's modulus of elasticity E. D^{α} denotes a fractional derivative, a derivative of generally noninteger order. In the case of the springpot, the order of the fractional derivative is specifically in the range $\alpha \in \langle 0; 1\rangle$. Setting $\alpha = 0$ the constitutive law becomes $\sigma(t) = E\varepsilon(t)$, the Hooke's law which describes purely elastic material. For $\alpha = 1$, we get $\sigma(t) = \eta\dot{\varepsilon}(t)$, which is Newton's equation of viscosity. From these limit cases it is obvious that the springpot itself behaves as a viscoelastic element somewhere between purely elastic and purely viscous.

2. Numerical approximation of fractional derivative

Sometimes, fractional derivative values can be simply computed analytically. However, when solving fractional differential equations, the analytical solution is often too difficult or even impossible to obtain. In

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these cases, numerical solutions are needed. To approximate the fractional derivative, Lubich (1986) introduces the following numerical scheme

$$D^{\alpha} f^{n} \approx \frac{1}{(\Delta t)^{\alpha}} \sum_{j=0}^{\mathrm{n}} w_j(\alpha) f^{n+1-j}, \tag{2}$$

where Δt is a size of the time step and j denotes the index of the step; $j = 0$ stands for the actual step at t^n while with increasing value of j we move retrospectively. The initial step at t^0 then corresponds to $j = n$. As the sum indicates, the numerical approximation of the fractional derivative depends on all the previous nodal values. An influence of the previous values is given by the weights $w_j(\alpha)$, which can be calculated as follows

$$w_0(\alpha) = 1, \qquad w_j(\alpha) = w_{\mathrm{j\text{-}1}}(\alpha)\frac{j-1-\alpha}{j}. \tag{3}$$

Setting the order of the derivative as $\alpha = 1$ the numerical scheme becomes the well-known scheme for the forward approximation of the first-order derivative.

When approximating an integer-order derivative, only a fixed and finite number of nodal values are needed for the approximation (e.g. the forward approximation of the first-order derivative uses two nodal values). In contrast, the fractional derivative approximation uses the nodal values of all previous steps, each of them multiplied by a corresponding weight. It is then obvious that the number of values included in the approximation increases with each additional step. Therefore, the computation of the approximated value in the next step always requires more effort with increasing computational time than is needed for the previous step.

2.1. Use for the springpot response analysis

Substitution of the numerical scheme for the fractional derivative, Eq. 2), into the constitutive law for springpot, Eq. (1), allows us to express the numerical approximation of the constitutive law for the springpot. It can be further adjusted, considering $w_0(\alpha) = 1$, to

$$\varepsilon^{n+1} = \frac{\sigma^n}{E}\left(\frac{\Delta t}{\tau_{\mathrm{c}}}\right)^{\alpha} - \sum_{j=1}^{\mathrm{n}} w_j(\alpha)\varepsilon^{n+1-j}. \tag{4}$$

which gives the numerical algorithm to express the response of the strain to the prescribed stress load.

3. Reduced approximation of the fractional derivative

The numerical algorithm presented by Eq. (4) includes a sum of all previous strain nodal values, each multiplied by the corresponding weight. As mentioned earlier, computing this sum is memory and computing time intensive. For this reason, fractional derivatives might seem unsuitable for use in large-scale numerical analyses, at least in this form of numerical approximation.

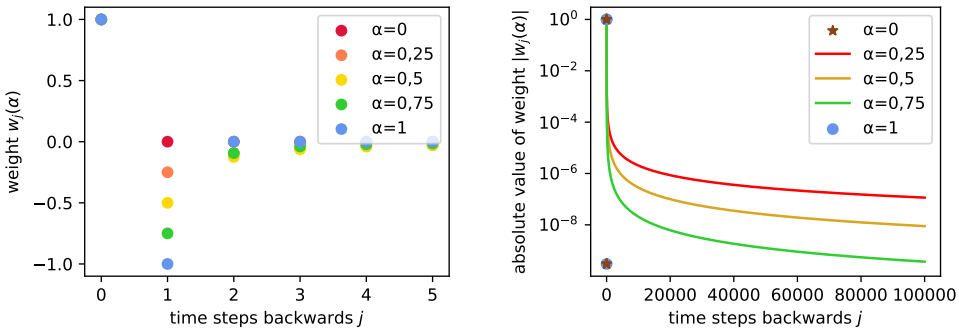

Fig. 1: Weight function for different values of parameter α

However, let us focus on the weights that are computed according to Eq. (3). The current value is multiplied by the weight $w_0(\alpha) = 1$. As we compute the weights, we see that they are decreasing (in absolute value),

and therefore the influence of the older nodal values is lower than the influence of the most recent ones, as shown in Fig. 1.

This leads us to an idea of whether it would be possible to adjust the approximation of Eq. (2) and reduce the number of nodal values used by the numerical scheme. Since the most recent values have the highest weights, the idea is to base the approximation on n the most recent values and somehow reduce the number of older ones.

3.1. Suggested reduced approximations

Several numerical experiments were performed, and in this section some of the performed reductions are presented. Note that the results presented correspond to the response derived for the prescribed Heaviside stress load (constant stress value acting from $t = 0$).

Approximation 1: The first suggested approximation reduces the number of adopted nodal values only to m of the most recent, while the remaining (older) values are not included in the approximation at all. In this case, the numerical algorithm, Eq. (4), is valid only for $n \leq m$ while for $n > m$ it can be rewritten as follows

$$\varepsilon^{n+1} = \frac{\sigma^n}{E}\left(\frac{\Delta t}{\tau_c}\right)^{\alpha} - \sum_{j=1}^{\mathrm{m}} w_j(\alpha)\varepsilon^{n+1-j}. \tag{5}$$

The resulting response is shown in the graphs in Fig. 2 by the orange line. The complete numerical solution according to Eq. (4) is depicted in the graphs by the black dashed line and the blue line represents the analytical solution (which can be obtained for this simple loading case). From the graph, it is obvious that this reduction is not suitable for the approximation of the material response.

Approximation 2: In this case the reduction also uses m of the most recent nodal values and therefore for $n \leq m$ the full algorithm, Eq. (4), is valid. Compared to the previous case, the unused $n - m$ values are included by their mean value multiplied by the sum of the remaining weights. This numerical algorithm can be written in the following form

$$\varepsilon^{n+1} = \frac{\sigma^n}{E}\left(\frac{\Delta t}{\tau_c}\right)^{\alpha} - \sum_{j=1}^{\mathrm{m}} w_j(\alpha)\varepsilon^{n+1-j} - \frac{1}{n-m}\sum_{j=\mathrm{m}+1}^{\mathrm{n}} \varepsilon^{n+1-j} \sum_{j=\mathrm{m}+1}^{\mathrm{n}} w_j(\alpha). \tag{6}$$

The response is shown in Fig. 2 by the green line with $m = 100$. The reduction also does not perfectly fit the exact solution. However, the result is much better than for the previous case and this reduction seems to have some potential for being used as an approximation.

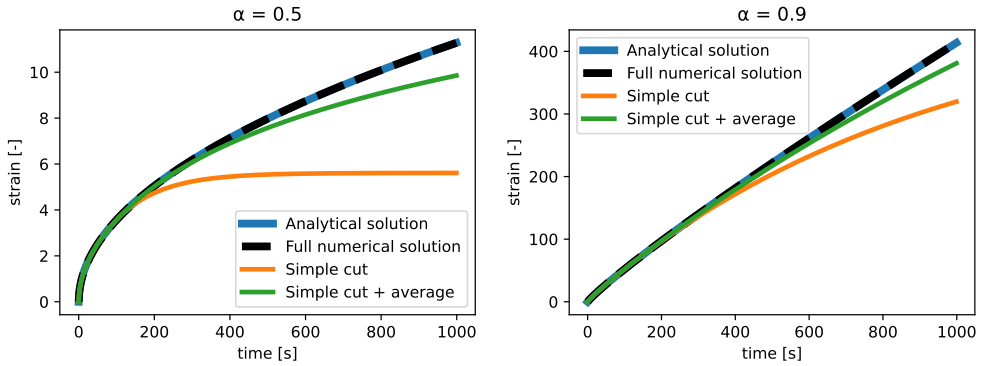

Fig. 2: Approximation for reduced number of considered steps $m = 100$

Approximation 3: This approximation was originally introduced by MacDonald et al. (2015) and is based on extending the time step by ignoring some of the older ones. The numerical scheme for computing the sum is shown in Fig. 3 – only the red steps are considered in the computation. As shown, in area $n - m$ (the most recent steps) all steps are considered; in area $m - p$ only each second step is considered, in area $p - q$ only each fourth is considered; the steps $q - 0$ at the beginning of the time row are considered in the usual way.

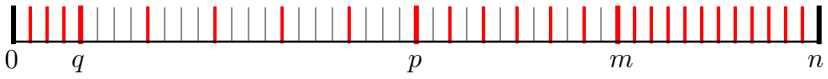

Fig. 3: Numerical scheme for the step-skipping algorithm

When skipping some of the steps, the weights also need to be adjusted by multiplying them by a suitable coefficient. The response for this approximation is shown in Fig. 3 for three different sets of multiplying coefficients. For the green line, the weights are kept unchanged; the blue line multiplies the weights by 2 in the area $m - p$ and by 4 in the area $p - q$. The orange line has an approach similar to that of the blue one, but the coefficients are further amplified to the power of α. As shown, all of the coefficient choices provide a pretty good approximation. However, for a different springpot parameter α, the optimal choice of coefficient seems to be different, and therefore further analysis is needed.

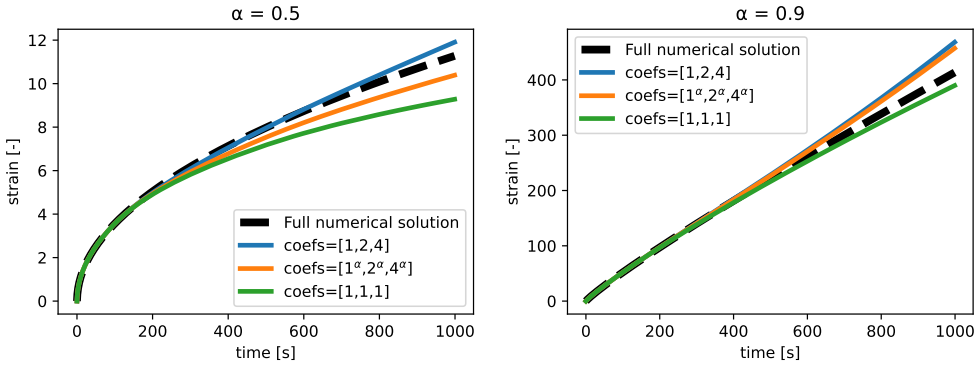

Fig. 4: Approximation with skipping some of the nodal values older than $m = 100$

4. Conclusions

Fractional calculus is a strong tool for describing viscoelastic materials. Although some of the problems are possible to describe analytically, solving fractional differential equations often cannot be done without the use of numerical algorithms. For standard integer-order derivatives, their numerical approximations includes only a finite constant number of nodal values. This is different for the fractional derivative because the numerical approximation includes all the previous nodal values, and each of them has a different weight for each subsequent time step. For this reason, it is more time and memory consuming.

This paper examines three possible modifications of the original numerical scheme to approximate the fractional derivative with a reduced number of nodal values. While the last two approaches suggest their potential, further investigation is needed, particularly in light of the application to both a simple loading condition and a rheological model.

Acknowledgments

The support of the Czech Science Foundation, grant No. 25-16071S, as well as the project SGS25/036/OHK1 / 1T/11, is gratefully acknowledged.

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