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FEM-BASED MODELING OF GENERALIZED AMPLITUDES IN STOCHASTIC VAN DER POL SYSTEMS

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Abstract: *This paper investigates various approaches for approximating the stationary aeroelastic response characteristics under near-resonance conditions, with a focus on the lock-in regime. A reduced form of the Fokker–Planck equation—derived via stochastic averaging—is used to represent the long-term behavior. Comparative analysis is performed between traditional finite element solutions and refined semi-analytical techniques based on a Galerkin-type expansion of an analytical solution available for exact resonance conditions. Although full verification via Monte Carlo simulations is constrained due to the elusive nature of the generalized partial amplitudes inherent in the reduced FPE framework, indicative comparisons reveal an unexpectedly analogy across methods. These results highlight both the practical value and the limitations of different modeling strategies in the probabilistic assessment of nonlinear systems.*

Keywords: Fokker-Planck equation, Stochastic Averaging, Galerkin approximation, Finite Element Method, Monte Carlo Simulation

1. Introduction

The dynamic behavior of slender engineering structures—particularly in aeroelastic applications—is often driven by a combination of periodic and random excitations. To capture such complex responses, simplified nonlinear models, such as a single-degree-of-freedom van der Pol-type oscillator, are frequently used. A central task in this analysis is determining the stationary probability distribution of the system's response, which typically involves solving the Fokker–Planck equation (FPE). This paper is part of ongoing research and builds on the concept of stationary partial amplitudes introduced by Náprstek et al. (2021), derived from the Itô solution of a stochastic van der Pol oscillator. That work focused on stationary responses in the lock-in region and presented an analytical solution to the reduced FPE in the case of exact resonance.

An extension of this approach across the full lock-in domain, using Galerkin approximations with polynomial basis functions, was later proposed in (Náprstek and Fischer, 2024). However, the accuracy of this method remains uncertain due to numerical issues in integrating higher-order polynomial corrections. Independent validation of the theoretical framework is difficult, primarily due to the abstract nature of the generalized averaged amplitudes. Nonetheless, an earlier attempt to compare numerical simulations with these amplitudes was reported by Fischer and Náprstek (2020). Though precise matching was challenging, the overall response amplitude showed visually good agreement.

This contribution introduces two new elements. First, it shows that the reduced FPE can be effectively solved using a standard FEM solver. Second, it carries out a time-domain stochastic simulation of the original van der Pol system, estimating the mean partial amplitudes from the sine and cosine components of the dominant Fourier mode.

2. Mathematical Model

This paper presents an extension and comparative evaluation of previously published results. Due to space limitations and editorial constraints on similarity, the reader is referred to (Náprstek et al., 2021; Náprstek and Fischer, 2024) for details of the mathematical model.

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The physical system under consideration is described by a SDOF van der Pol-type oscillator:

$$\ddot{u} - (\eta - \nu u^2)\dot{u} + \omega_0^2 u = P\omega^2 \cos\omega t + h\xi(t)\,, \tag{1}$$

where: $u, v = \dot{u}$ are the displacement $[m]$ and velocity $[ms^{-1}]$; η, ν are the parameters of the linear and quadratic damping, respectively $[s^{-1}, s^{-1}m^{-2}]$; ω_0, ω are the eigen-frequency of the linear SDOF system and frequency of the vortex shedding $[s^{-1}]$; $f(t)$ represents external excitation: $f(t) = P\omega^2\cos\omega t + h\xi(t)$; $P\omega^2$ and $\xi(t)$ are the amplitude of the harmonic excitation force $[ms^{-2}]$ and the broadband Gaussian random process $[1]$; and h is the multiplicative constant $[ms^{-2}]$.

To facilitate the application of the stochastic averaging method (Roberts and Spanos, 1986; Bernard, 2003), the displacement and velocity are expressed using a first harmonic approximation as follows:

$$u(t) = a_c\cos\omega t + a_s\sin\omega t\,,\ v(t) = -a_c\omega\sin\omega t + a_s\omega\cos\omega t\,,\ \text{where}\ \dot{a}_c\cos\omega t + \dot{a}_s\sin\omega t = 0\,. \tag{2}$$

By employing the harmonic balance technique along with stochastic averaging in the sense of Itô calculus, the FPE can be derived for the probability density function (PDF) $p(a_c, a_s)$ of the slowly varying, stochastically averaged amplitudes. Approximation and estimation of these random quantities and their PDF is the main subject of this contribution.

3. Numerical Solution

The numerical analysis of the system is based on two complementary approaches: direct time-domain simulation of the original van der Pol (VDP) system, and numerical solution of the FPE, subjected to either zero or zero-flux boundary conditions at infinity. In the stationary regime—an essential condition for applying stochastic averaging—the time derivative in the FPE vanishes, resulting in a reduced form of the FPE. This reduced equation can be solved analytically or semi-analytically in several specific scenarios, as demonstrated by Lin and Cai (1988). In the case of exact resonance, $\Delta = (\omega_0^2 - \omega^2)/(2\omega) = 0$, the closed-form solution $p_0(a_c, a_s)$ was derived by Náprstek et al. (2021).

All approaches are illustrated using comparable plots for direct visual comparison. Each figure contains four plot pairs arranged in a 2×2 grid. Each pair includes a contour plot of the joint PDF $p(a_c, a_s)$ (left) and corresponding marginal probability density curves (right), computed for several fixed values of a_s (left column) or a_c (right column). The top row presents results for exact resonance, while the bottom row corresponds to a significantly detuned case, $\Delta = 0.125$, which lies just inside the lock-in region boundary. The following parameter values are used : $\eta = \frac{1}{2}$, $\nu = \frac{1}{4}$, $\omega_0 = 1$, $P = 1$, $S = 2$.

3.1. Exact Solution in the Resonance and its Galerkin-type Extension

In the stationary case, implied by the usage of stochastic averaging, the cross-probabitiliy of partial amplitudes is governed by the reduced FPE with zero boundary conditions at the infinity.

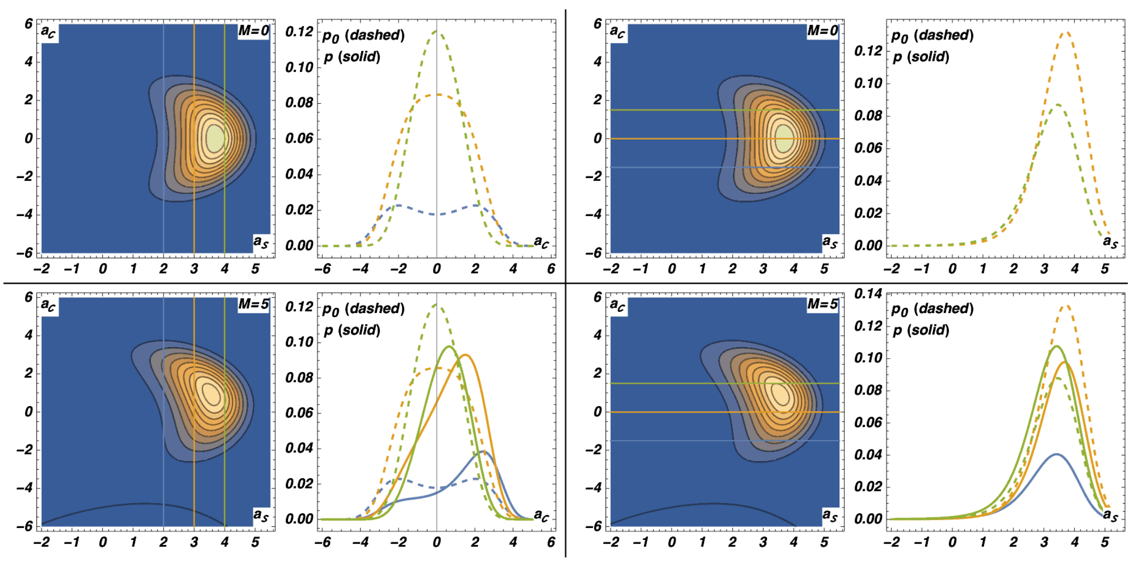

Fig. 1: Analytical solution for $\Delta = 0$ (top row) and Galerkin approximation for $\Delta = 0.125$ (bottom row). Left column: marginal densities for fixed values $a_c = \{2, 3, 4\}$; right column: dtto for $a_s = \{-\frac{3}{2}, 0, \frac{3}{2}\}$.

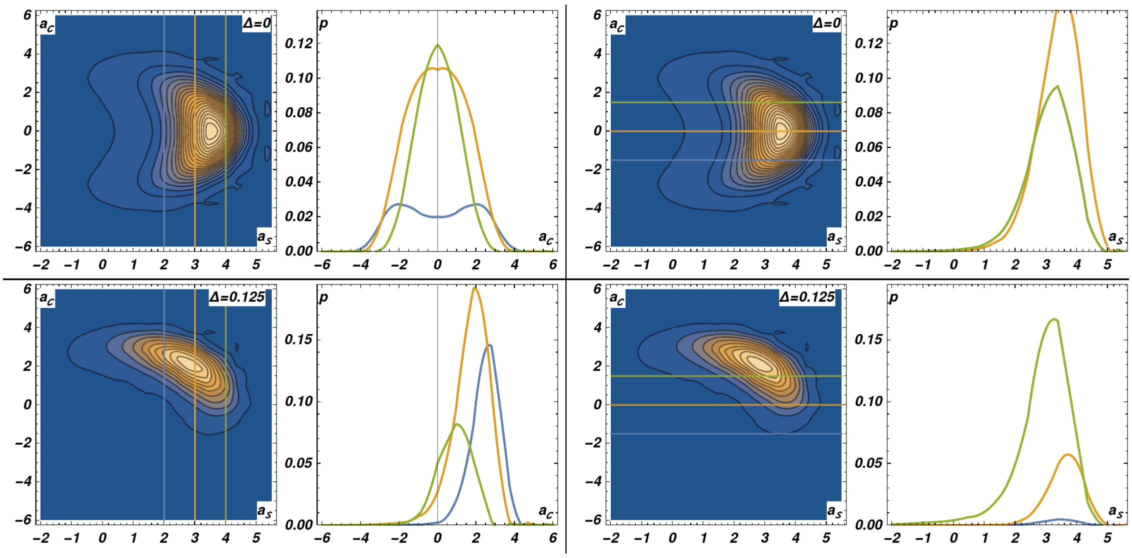

Fig. 2: FEM solution for the resonant case (top row) and for $\Delta = 0.125$ (bottom row). Left column: marginal densities for fixed values $a_c = \{2, 3, 4\}$; right column: dtto for $a_s = \{-\frac{3}{2}, 0, \frac{3}{2}\}$..

To extend the solution to cases with nonzero detuning, a Galerkin-type expansion is employed. The unknown joint PDF is approximated as

$$p(a_c, a_s) = p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} \cdot a_c^{k-l} \cdot a_s^l \,, \tag{3}$$

where $p_0(a_c, a_s)$ is a weight function taken as the exact solution for $\Delta = 0$, and the coefficients q_{kl} represent higher-order corrections.

Figure 1 illustrates this method: the top row shows the analytical stationary solution for $\Delta = 0$ (i.e., $M = 0$), while the bottom row shows the result for $\Delta = 0.125$ using a correction up to $M = 5$ polynomial terms. This correction level corresponds to the inclusion of five stochastic moments, as interpreted in the framework of (Náprstek and Fischer, 2024).

3.2. Finite Element-Based Solution

Unlike the specialized solvers required for high-dimensional problems (see e.g., Král and Náprstek (2017)), here a straightforward, built-in FEM solver from Mathematica version 12.3 is employed. Because the homogeneous PDE admits infinitely many solutions—and the FEM solver by default yields the trivial zero solution—an additional constraint is required to extract a meaningful non-zero solution.

This constraint is introduced by first solving a reduced 1D boundary value problem (BVP). Fixing, e.g., $a_c = 0$, the resulting ODE is solved with a prescribed zero value on one boundary and a non-zero value in the region expected to contain the mode of the distribution. This can be achieved either by solving two BVPs or, if the solver allows, by specifying one boundary condition and one interior condition. Inadequate decay at the boundary or mismatch at the interior point indicates an improperly chosen domain size.

The resulting 1D profile is then imposed as a boundary condition along a shared edge in a split domain strategy for solving the full 2D problem. Alternative formulations include adding integral constraints or using Lagrange multipliers, but the chosen method maximizes simplicity and computational availability.

Figure 2 shows the FEM solution. The results display visually good agreement with the analytical solutions shown in Fig. 1. However, it is evident that the Galerkin correction with $M = 5$ moments may not be sufficient to fully capture the effects of detuning near the boundary of the lock-in region.

3.3. Monte Carlo Simulations

The Monte Carlo simulation of the original van der Pol system Eq. (1) was carried out under combined harmonic and stochastic excitation. From the steady-state portion of each realization, the FFT was applied to extract the dominant frequency component. The corresponding amplitude was then decomposed into sine and cosine contributions, interpreted as estimates of the partial response amplitudes a_c and a_s, respectively. Their absolute values were used to construct a two-dimensional smoothed and normalized histogram.

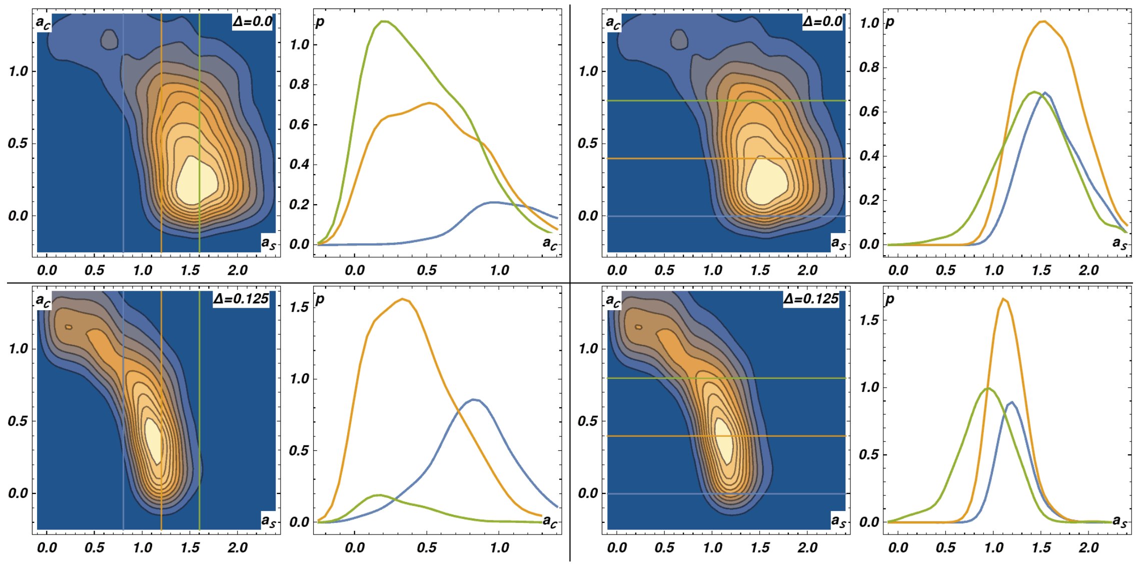

Fig. 3: MC-simulation approximation of the averaged partial amplitudes for $\Delta = 0$ (top) and $\Delta = 0.125$. Left column: marginal densities for values $a_s = \{0.8, 1.2, 1.6\}$; right column: dtto for $a_c = \{0, 0.4, 0.8\}$.

While this approach does not retain the sign and absolute value of information of the amplitudes, the resulting spatial distribution, shown in Fig. 3, exhibits a qualitative agreement with the Galerkin and FEM-based solutions. A potential refinement for future work would involve cycle-by-cycle trigonometric fitting of the time series, enabling a more accurate reconstruction of amplitude statistics.

4. Conclusions

This paper presents three complementary approaches to evaluating stationary responses in nonlinear stochastic systems: a simple FEM-based solution of the reduced Fokker–Planck equation using `NDSolve`, a Galerkin-type approximation of the Fokker–Planck equation, and a time-domain simulation estimating partial amplitudes from dominant Fourier components. While the time-domain method does not preserve absolute amplitudes, its relative distribution aligns well with the FEM results and offers a basis for future refinements, such as cycle-by-cycle trigonometric fitting. The Galerkin-type method provides theoretical advantages in terms of reduced-order modelling and basis function interpretation; however, it fails to accurately reproduce the probability density function shape when compared to the FEM benchmark, indicating the need for further development or adaptive basis enhancement.

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