31st International Conference
ENGINEERING MECHANICS 2025
Medlov, Czech Republic, May 12 –14, 2025
Paper **#59**, pp. 137–140, doi: 10.21495/em2025-137

NUMERICAL SIMULATION OF TRANSIENT AEROELASTIC PHENOMENA INFLUENCED BY STOCHASTIC RESONANCE IN CROSS FLOW CONDITIONS

Náprstek J. [1], **Fischer C.** [1]

Abstract: *Stochastic resonance is a phenomenon observed in nonlinear dynamical systems with bistable behavior under combined deterministic and stochastic excitation. In its classical form, it is modeled using a Duffing-type oscillator subjected to harmonic forcing and additive white noise, resulting in noise-assisted transitions between stable states. This study investigates numerical methods for capturing transient effects related to stochastic resonance in the context of aeroelastic post-critical behavior, with a focus on a prismatic beam exposed to cross air flow. Motivated by wind tunnel observations, stochastic resonance is explored as a theoretical framework for modeling complex aeroelastic responses in bridge decks. Two computational approaches are employed: direct stochastic simulation using Monte Carlo methods, and numerical solution of the associated Fokker–Planck equation using the method of lines. The results highlight the capability of these methods to reproduce resonance-driven switching and demonstrate their potential for predicting transient aeroelastic responses relevant to the stability of slender structures under aerodynamic loading.*

Keywords: Stochastic Resonance, Aeroelasticity, Fokker-Planck Equation, Duffing Oscillator

1. Introduction

Stochastic resonance (SR) is a nonlinear dynamical phenomenon in which random noise contributes constructively to a system's response when combined with periodic deterministic excitation. First noted in studies of Brownian motion by Kramers (1940) and later formalized in climatological contexts by Nicolis (1981, 1982), SR has since found applications across diverse fields including statistical physics, signal processing, and more recently, structural mechanics.

In engineering dynamics, SR is of particular interest due to its ability to influence the post-critical behavior of nonlinear systems. It typically manifests in bistable systems governed by nonlinear restoring forces, such as the Duffing oscillator with negative linear stiffness. In such systems, a combination of low-amplitude harmonic excitation and Gaussian white noise can induce periodic transitions—so-called "interwell hopping"—between stable equilibria, leading to an amplified response at an optimal noise intensity.

Applications in mechanics, and especially in aeroelasticity, have only recently begun to consider stochastic resonance as a viable modeling framework. Slender structures such as beams, bridge decks, towers, or cables exposed to cross air flow may exhibit complex post-critical dynamics, including divergence, flutter, buffeting, and switching between these modes. These behaviors are often highly sensitive to small perturbations, and stochastic resonance offers a useful framework for analyzing the influence of environmental noise—such as turbulence or gusts—on such systems. When stochastic components of wind interact with periodic effects like vortex shedding or oscillatory aerodynamic forces, resonance conditions may arise, amplifying vibrations near instability thresholds. This amplification can be critical, particularly in civil engineering structures with high flexibility and low damping. For further background and a broader treatment of the topic, see (Náprstek, 2015).

This paper investigates the numerical modeling of SR-induced transient effects in a simplified aeroelastic system representing a prismatic beam in cross flow. The system is modeled as a single-degree-of-freedom

[1] Ing. Jiří Náprstek, DrSc., RNDr. Cyril Fischer, Ph.D., Institute of Theoretical and Applied Mechanics, Prosecká 76, 190 00 Prague 9, tel. +420 225 443 221, e-mail {naprstek,fischerc}@itam.cas.cz

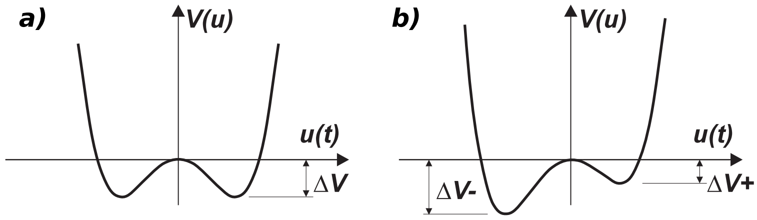

Fig. 1: a) symmetric potential, b) asymmetric potential

Duffing oscillator subjected to combined harmonic and stochastic forcing. The associated Fokker–Planck equation (FPE), which governs the evolution of the system's probability density function, is solved numerically using a straightforward finite element implementation available in Mathematica. Unlike the advanced finite element method (FEM) techniques developed particularly for FPE by Náprstek and Král (2014), the present work adopts a readily accessible commercial platform to demonstrate the feasibility of capturing essential resonance-driven behavior. In addition, direct stochastic simulation via the Monte Carlo (MC) method is employed to support and verify the qualitative features observed in the FPE solution.

2. Mathematical Model

The classical formulation of SR is typically based on the Duffing oscillator with a negative linear stiffness component. This represents the simplest archetype of SR, where the system is excited by a combination of a deterministic harmonic force with fixed frequency and additive Gaussian white noise. In normalized form, the nonlinear unit-mass single-degree-of-freedom (SDOF) oscillator is expressed as:

$$\dot{u} = v, \tag{1}$$

$$\dot{v} = -2\omega_b v - V'(u) + P(t) + \xi(t)\,, \tag{2}$$

where $V(u)$ is the potential function that yields the Duffing-type restoring force, given by:

$$V(u) = -\frac{\omega_0^2}{2}u^2 + \frac{\gamma^4}{4}u^4 \qquad \Rightarrow \qquad V'(u) = \frac{\mathrm{d}V(u)}{\mathrm{d}u} = -\omega_0^2 u + \gamma^4 u^3\,. \tag{3}$$

Here, ω_0 and ω_b denote the natural circular frequency and damping frequency of the corresponding linear system, respectively. The negative linear stiffness in $V'(u)$ renders the equilibrium at the origin metastable, while the cubic nonlinear term acts as a restoring force for large displacements, forming a bistable potential. The term $\xi(t)$ represents Gaussian white noise of intensity $2\sigma^2$, satisfying the statistical conditions:

$$\mathrm{E}\{\xi(t)\} = 0, \quad \mathrm{E}\{\xi(t)\xi(t')\} = 2\sigma^2\delta(t-t'), \tag{4}$$

where $\mathrm{E}\{\bullet\}$ denotes the expectation operator and $\delta(t)$ is the Dirac delta function. The external excitation is defined as a harmonic force per unit mass, with P_0 denoting the amplitude and Ω the excitation frequency:

$$P(t) = P_0 \cos \Omega t\,. \tag{5}$$

Figure 1 illustrates two possible configurations of the system: (a) a symmetric potential, in which the energy required to transition between wells is equal in both directions; and (b) an asymmetric potential modified by an additional linear spring. The latter configuration can, if the spring is sufficiently stiff, drive the system toward monostability. Notably, even such monostable systems can exhibit stochastic resonance under suitable excitation conditions. Equation (3) defines a symmetric potential with two stable equilibria at $u_{min} = \pm\omega_0/\gamma^2$ and one unstable equilibrium (a saddle point) at $u_{max} = 0$, which represents the potential barrier between the wells.

In a bistable system subject to stochastic fluctuations, the Kramers rate r_K describes the average rate at which noise induces transitions between the two potential wells. Stochastic resonance occurs when the average noise-induced transition time is approximately equal to half the period of the external forcing. Accordingly, an estimate of the optimal noise intensity for enhancing stochastic resonance can be expressed as σ_{SR}^2 (ΔV is the height of the barrier, cf. Fig. 1):

$$r_K = \frac{\omega_0}{2\pi}\sqrt{2}\cdot\exp\left(-\frac{\Delta V}{\sigma^2}\right), \qquad \sigma_{SR}^2 \approx \Delta V\left(\log\frac{2\sqrt{2}\,\omega_0}{\Omega}\right)^{-1}, \qquad \Delta V = \frac{\omega_0^4}{4\gamma^4}\,. \tag{6}$$

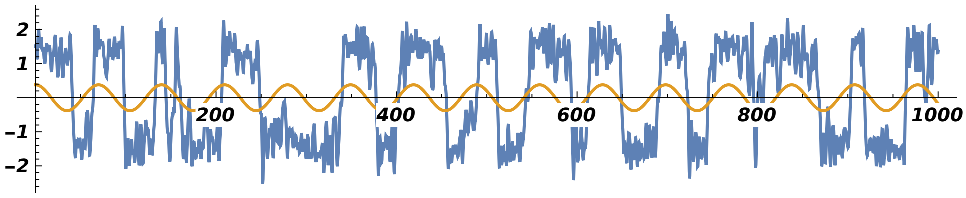

Fig. 2: Input (yellow) and response (blue) of the stochastic Duffing equation for $\sigma = 0.4$.

3. Numerical Analysis of Stochastic Resonance in the Duffing Oscillator

The SR phenomenon in the Duffing oscillator subject to additive noise and periodic forcing has been investigated numerically using two complementary approaches. The first is a Monte Carlo simulation of the underlying stochastic differential equation using the Euler–Maruyama method. The second is a FEM solution of the corresponding FPE, formulated as a PDE in the phase space variables (u, v).

Both approaches are used here for illustrative purposes, with the goal of demonstrating that the applied numerical methods are capable of capturing the transient dynamics of the system, such as noise-induced interwell transitions characteristic of the stochastic resonance phenomenon. The examples are not intended for high-precision quantitative validation, but rather to show that the methods employed can qualitatively resolve the key features of resonance-driven switching behavior.

In all cases, the system was initialized in the right-hand (positive) well, with deterministic initial conditions $(u, v) = (\sqrt{2}, 0)$ for the MC simulation, and a narrow bivariate normal probability density centered at the same point for the FEM solution. A representative time history of the Duffing trajectory under noise intensity $\sigma = 0.4$ is shown in Fig. 2, where resonance-like behavior is visible in the synchronized well-switching. The parameters used in the examples are as follows: $\omega_0^2 = 1$, $\gamma^4 = 0.5$, $\omega_b = 0.25$, $P_0 = 0.38$, $\Omega = 0.09$. The corresponding $\sigma_{SR} \approx 0.38$.

3.1. Monte Carlo Simulation

The stochastic Duffing equation was integrated numerically using the Euler–Maruyama method over an ensemble of 50 000 independent realizations. The resulting trajectories were used to construct time-evolving estimates of the bivariate probability distribution $p(u, v, t)$, with particular focus on the marginal distribution in u. The left part of Fig. 3 shows the evolution of this marginal distribution in axonometric view, starting from the localized initial condition and covering more than one complete forcing period.

The probability density was observed to oscillate between the wells in a manner synchronized with the external forcing, indicating the presence of stochastic resonance, see also Fig. 2. The switching frequency matched approximately half the forcing frequency, consistent with the resonance condition derived from Kramers' rate. The transition behavior appeared sharply localized within the potential wells due to the finite ensemble size and the deterministic initial configuration.

3.2. Finite Element Solution of the Fokker–Planck Equation

The associated FPE for the probability density function $p(u, v, t)$ in the Duffing oscillator phase space is

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial u} - \frac{\partial}{\partial v}\left(\left(2\omega_b v + V'(u) - P(t)\right) p\right) = \sigma^2 \frac{\partial^2 p}{\partial v^2}\,. \tag{7}$$

Equation (7) was solved numerically using Mathematica's built-in finite element method. The computational domain was chosen large enough to capture the region of effectively nonzero probability throughout the simulation. The initial condition was defined as a bivariate Gaussian centered at $(\sqrt{2}, 0)$, aligned with the deterministic initial state used in the MC analysis.

In principle, no-flux (Neumann) boundary conditions would be appropriate for this type of parabolic equation, as they ensure conservation of total probability. However, it was observed that Mathematica's default FE solver did not handle Neumann conditions robustly in this setting. Thus, Dirichlet boundary conditions were imposed, setting the solution to zero at the domain boundaries. This inevitably introduced artificial probability loss over time—approximately 40% over a single unit of time in this case. To compensate for this effect, the solution was rescaled at regular time intervals to restore the total probability to unity.

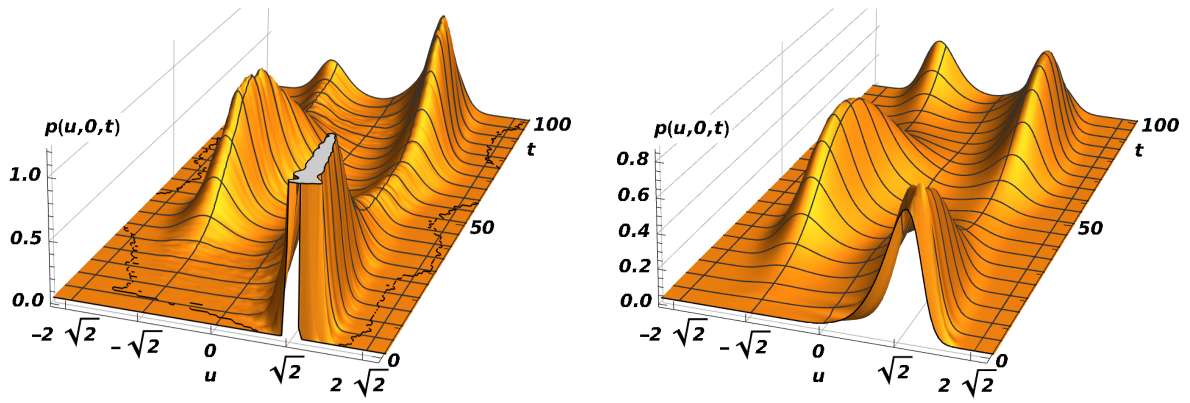

Fig. 3: Time evolution of the marginal probability density in the u-coordinate obtained from Monte Carlo simulation (left) and finite element solution of the Fokker–Planck equation (right).

The right part of Fig. 3 presents the evolution of the marginal probability density in u over time. While the general trend of alternating dominance between the wells is still visible, the profiles appear flatter and less sharply peaked than in the MC case. This can be attributed to numerical diffusion and the lack of precise mass conservation inherent in the standard FEM implementation. It should be emphasized that the current FEM results are not intended as high-fidelity solutions. For more accurate modeling, the use of specialized computational fluid dynamics (CFD) solvers designed for parabolic PDEs is advisable, cf., e.g., (Bieterman and Babuška, 1982). In particular, implementation of mass-preserving boundary treatments and stabilization techniques would be essential to improve solution quality.

4. Conclusions

The occurrence of stochastic resonance in the Duffing oscillator has been investigated using two numerical approaches: stochastic simulation via the Euler–Maruyama method, and FEM of the associated Fokker–Planck equation. Both methods were shown to qualitatively capture the transient switching behavior between potential wells in the near-resonant regime. In particular, the synchronization between noise-induced transitions and the periodic forcing was clearly visible in the evolution of the probability density.

While the results illustrate the capability of both approaches to detect resonance effects, the analysis remains preliminary. Further refinement, including the use of advanced numerical schemes with mass-conserving properties and quantitative evaluation of transition rates, is required for a more comprehensive understanding. The topic warrants deeper exploration, particularly with regard to parameter sensitivity, spectral amplification, and the role of boundary conditions in the probabilistic framework.

Acknowledgments

The kind support of Czech Science Foundation project No. 24-13061S and of the RVO 68378297 institutional support are gratefully acknowledged. The authors used the ChatGPT language model (OpenAI) to assist with language polishing and proofreading; all suggestions were reviewed and revised to preserve the intended meaning.

References

Bieterman, M. and Babuška, I. (1982) The finite element method for parabolic equations. *Numerische Mathematik*, 40, 3, pp. 339–371.

Kramers, H. A. (1940) Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, VII, 4, pp. 284–304.

Nicolis, C. (1981) Solar variability and stochastic effects on climate. *Solar Physics*, 74, pp. 473–478.

Nicolis, C. (1982) Stochastic aspects of climatic transitions-response to a periodic forcing. *Tellus*, 34, pp. 1–9.

Náprstek, J. (2015) Stochastic resonance: Challenges to engineering dynamics. *Computational Technology Reviews*, 12, pp. 53–101.

Náprstek, J. and Král, R. (2014) Finite element method analysis of Fokker-Planck equation in stationary and evolutionary versions. *Advances in Engineering Software*, 72, pp. 28–38.