31st International Conference
ENGINEERING MECHANICS 2025
Medlov, Czech Republic, May 12 –14, 2025
Paper **#42**, pp. 13–18, doi: 10.21495/em2025-13

APPLICATIONS OF GEOMETRIC FLOWS

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Abstract: *Geometric flows describe the evolution of shapes driven by geometric quantities such as curvature. While originating in pure mathematics, where they played a central role in milestones like the proof of the Poincaré conjecture, these flows also have a rich history of applications across science and engineering. In this paper, we provide a brief overview of the key ideas behind geometric flows, focusing on curve and surface evolution in Euclidean space, and highlight their relevance to modeling dislocation lines in crystals, vortex filaments in fluids, and optimal structural forms in engineering. Our aim is to provide a concise but self-contained exposition, emphasizing the conceptual beauty and broad utility of this mathematical framework.*

Keywords: geometric flows, curve evolution, surface optimization, applications in mechanics

1. Introduction and Scope

Geometric flows describe the evolution of shapes driven by curvature and other geometric quantities. Though originating in differential geometry, they now arise across disciplines—modeling filaments in fluids, optimizing architectural forms, and simulating biological membranes. Their strength lies in combining physical fidelity with geometric clarity, often leading to elegant and computationally efficient formulations.

This paper aims to provide a concise and self-contained overview of geometric flows, with emphasis on the extrinsic evolution of curves and surfaces in three-dimensional Euclidean space. We highlight the central mathematical ideas and demonstrate their utility through concrete examples drawn from science and engineering. Beyond exposition, this paper aims to inspire applied scientists and engineers to consider geometric flows as part of their modeling toolkit. When problems involve evolving shapes, curvature, or spatial constraints, these flows offer a natural, elegant, and perhaps underused framework.

Structure of the paper. Section 2. introduces the general theory of geometric flows, distinguishing intrinsic and extrinsic types. Section 3. focuses on curve flows: we present the geometric framework, key examples, and applications across physics, biology, and computer graphics. Section 4. treats surface evolution, from mean curvature flow to Willmore flow, with illustrative applications in materials science, visualization, and structural design. We conclude in Section 5. with an outlook on the future role of geometric flows in applied mathematics and computational modeling.

2. Overview of Geometric Flows

Geometric flows are partial differential equations that govern the evolution of geometric objects—curves, surfaces, or higher-dimensional manifolds—according to geometric quantities such as curvature.

They are typically categorized as intrinsic or extrinsic. Intrinsic flows evolve the internal geometry of a manifold. A canonical example is the Ricci flow, where the Riemannian metric g evolves by

$$\partial_t g = -2\mathrm{Ric}(g),$$

driven by the Ricci curvature tensor. Such flows have led to profound advances in geometry and topology, including the proof of the Geometrization and Poincaré conjecture from Perelman (2003).

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Extrinsic flows, by contrast, govern the evolution of a submanifold immersed in an ambient space. Formally, let $\mathcal{M}$ be a smooth manifold and $(\mathcal{N}, \overline{g})$ a Riemannian ambient manifold. A time-dependent immersion $F : \mathcal{M} \times [0, \underline{t}) \to \mathcal{N}$ evolves by an extrinsic geometric flow if

$$\partial_t F(p,t) = \upsilon(p,t),$$

where υ is a velocity vector field defined in terms of geometric quantities—such as curvature, torsion, or higher-order invariants—of the immersed manifold.

This work focuses on extrinsic flows, where the ambient space is Euclidean 3-space and the evolving manifold is a curve or a surface. Among various representation methods (e.g. level set or phase field), we adopt the parametric approach due to its mathematical clarity, direct physical interpretation, and numerical simplicity. We begin by exploring the evolution of space curves, then turn to the case of surfaces.

3. Geometric Flows of Space Curves

Let $\{\Gamma_t\}_{t\in[0,\underline{t})}$ be a family of closed curves in $\mathbb{R}^3$, each represented by a smooth map $\gamma(\cdot,t) : S^1 \to \mathbb{R}^3$, where $u \in S^1$ is a local parameter. The local rate of parametrization is $g(u,t) := |\partial_u \gamma(u,t)|$, and the arc-length element is $\mathrm{d}s = g(u,t), \mathrm{d}u$. The Frenet frame consists of the unit tangent $T = \partial_s \gamma$, normal $N = \partial_s T / |\partial_s T|$, and binormal $B = T \times N$, with associated curvature $\kappa = |\partial_s T|$ and torsion $\tau = -\langle \partial_s B, N\rangle$. A general geometric flow of the curve Γ_t takes the form of the following initial-value problem:

$$\partial_t \gamma = \upsilon_T T + \upsilon_N N + \upsilon_B B \qquad \text{in } S^1 \times (0, \underline{t}),$$

$$\gamma|_{t=0} = \gamma_0 \qquad \text{in } S^1,$$

where γ_0 is the initial parametrization and $\upsilon_T, \upsilon_N, \upsilon_B$ are scalar functions depending on geometric quantities such as κ and τ. Since tangential motion does not affect the curve's shape, we often set $\upsilon_T = 0$.

3.1. Examples of Curve Flows

Curve Shortening Flow: Arguably the most fundamental and widely studied geometric flow. It evolves curves by moving each point in the direction of the unit normal with speed equal to the curvature:

$$\partial_t \gamma = \kappa N.$$

Formally, this is the L^2-gradient flow of the curve length $E[\gamma] = \int \mathrm{d}s$, and can be viewed as a nonlinear analog of the heat equation: since $\kappa N = \partial_s^2 \gamma$, the evolution is parabolic up to reparametrization. Despite the nonlinearity introduced by arc-length derivatives, many classical tools apply. This inclides the maximum principle, comparison principles, and well-posedness theory.

The flow smooths out all initial roughness: any initial curve becomes analytic for $t > 0$, and singularities can only occur via curvature blow-up. In the planar case, Grayson's theorem (Grayson, 1987) states that any embedded curve remains embedded, becomes convex in finite time, and ultimately shrinks to a round point. In three dimensions, the dynamics are more intricate; self-intersections can develop and the asymptotics remain a topic of ongoing research (Minarčík and Beneš, 2020).

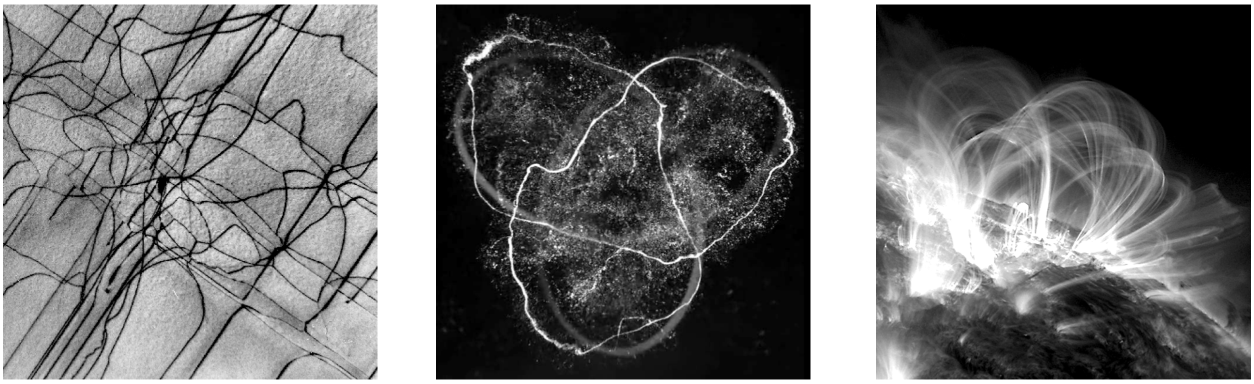
Due to its parabolic nature and strong smoothing character, curve shortening flow is a natural model for any application requiring regularization of noisy one-dimensional data.

Binormal Flow (Vortex Filament Equation): Like curve shortening flow, this evolution also uses the curvature magnitude as the speed, but moves in the binormal direction instead of the normal:

$$\partial_t \gamma = \kappa B = \partial_s \gamma \times \partial_s^2 \gamma.$$

This subtle shift leads to radically different behavior: the flow is Hamiltonian, not dissipative. It conserves arc length, total bending energy, and more, and corresponds to a geometric realization of the Schrödinger equation via the Hasimoto transform (Hasimoto, 1972).

Its evolution is highly intuitive: a circle simply translates along its binormal direction with constant speed, and more generally, any elastica evolves by a rigid transformation. For example, an elastic lemniscate rotates about its symmetry axis. As the name suggests, this flow arises in fluid dynamics as a simple model for vortex filament evolution (Ricca, 1991) and its arclength-conserving flows are beneficial for mesh-based implementations and geometric algorithms.



(a) Dislocation lines from Kacher and Robertson (2012).

(b) Knotted vortex from Kleckner and Irvine (2013).

(c) Solar magnetic field lines from Aschwanden et al. (2019).

Fig. 1: Examples of important natural phenomena at three drastically different scales involving one dimensional filaments that can be modeled as geometric flows of space curves.

Elastic Flow: Another classical motion law is the elastic flow, which arises as the gradient flow of the bending energy $E[\gamma] = \int \kappa^2 \mathrm{d}s$. The corresponding stationary curves, called elasticae, were first analyzed in $\mathbb{R}^2$ by Euler. The full evolution equation in $\mathbb{R}^3$ is more involved:

$$\partial_t \gamma = \left(2\kappa\tau^2 - \kappa^3 - 2\partial_s^2 \kappa\right) N - \left(4\partial_s \kappa \tau + 2\kappa \partial_s \tau\right) B,$$

This flow drives toward elasticae, which appear in applications ranging from architectural and mechanical design to computer graphics, where they serve in stylized shape generation, smooth interpolation, and realistic filament simulation. However, as a 4th-order flow, it poses greater analytical and numerical challenges.

Repulsive Flows (O'Hara-type Energies): More complex examples of gradient flows arise from repulsive energies, which regularize curves while preventing self-intersections. Such example is the Möbius energy,

$$E[\gamma] = \iint_{S^1 \times S^1} \left[\frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{\rho(u,v)^2} \right] \mathrm{d}u, \mathrm{d}v,$$

where $\rho(u, v)$ is the intrinsic distance along the curve. These flows preserve embeddings, making them suitable for modeling knotted structures, filament packing, and constrained deformations.

Repulsive energies have found use in knot theory, mathematical visualization, and surface design, particularly in contexts where self-avoidance is essential. Due to their nonlocality, the flows exhibit long-range interactions and often require careful preconditioning or hierarchical methods to be computationally tractable.

Minimal Surface Generating Flow: Final example from Minarčík and Beneš (2022) is given by $\partial_t \gamma = \tau^{-1/2} N$ and drives the curve to sweep out a minimal surface as its trajectory. Though highly singular at torsion-zero points, it provides a direct method for generating minimal surfaces with prescribed boundary.

3.2. Applications of Curve Flows

Many physical, biological, and engineered systems exhibit behavior that can be effectively captured by the evolution of one-dimensional filaments. These include dislocation lines in crystals, vortex filaments in fluids, magnetic flux tubes in astrophysical plasmas, and scroll waves in excitable media. Modeling such phenomena using geometric flows allows for reduced dimensionality, improved computational efficiency, and often, leads to better understanding and new insights.

In materials science, dislocation lines represent defects in the crystal lattice that mediate plastic deformation. Their curvature-driven motion can be modeled using geometric flows like curve shortening with additional force term (Mura, 1987). At much larger scales, similar filamentary structures appear in astrophysics, where solar magnetic field lines form dynamically evolving tubes resembling knotted space curves (Figure 1, Yeates et al. (2010)). Modeling such systems with geometric flows provides a unifying language for understanding local curvature effects and global topology alike.

In fluid dynamics, vortex filaments carry concentrated vorticity and evolve according to the binormal flow, capturing phenomena such as traveling waves, reconnection events, and knotted vortices observed in laboratory experiments (Figure 1). In quantum fluids, related dynamics emerge from Gross–Pitaevskii models

(a) Knotted DNA molecule from Wasserman et al. (1985).

(b) Meandering river bed from from Google Maps (2014).

(c) B.–Z. reaction scroll waves from Winkler et al. (2009).

Fig. 2: Besides physics, modelling of one-dimensional filaments is useful in many domains of science. This figure illustrates examples from biology, geology and chemistry.

(Barenghi et al., 1997; Zuccher and Ricca, 2022). The arc-length-preserving nature of the flow is also advantageous in discrete differential geometry, where it supports mesh-based simulations that respect geometric invariants. Similar ideas extend to excitable biological and chemical media: scroll waves in cardiac tissue or the Belousov–Zhabotinsky reaction propagate along filaments that evolve by geometric rules involving curvature and torsion (Maucher and Sutcliffe, 2016, 2019; Keener, 1988).

Natural curve evolution plays a role in Earth sciences as well, see Figure 2. Meandering rivers evolve through curvature-dependent erosion and deposition, and geometric models have been proposed that even incorporate nonlocal effects to account for upstream memory (Furbish, 1991). In computational settings, geometric flows have found wide use in graphics and animation, especially for simulating hair strands, ropes, and cables Chern et al. (2016). Elastic and repulsive flows offer physically plausible yet controllable behavior, enabling stylized and stable motion for interactive and rendered scenes (Yu et al., 2021).

Finally, geometric flows provide valuable tools for data analysis and signal processing. The parabolic nature of curve shortening flow, in particular, makes it a powerful method for denoising spatial trajectories, contours, or time-series data. It smooths noise while preserving essential geometric features, making it well suited for applications in computer vision, medical imaging, and robotics (Alexa et al., 2003; Beneš et al., 2004). Whether in physical modeling or data-driven contexts, curve flows offer a robust and geometrically grounded approach to the evolution of shape.

4. Geometric Flows of Surfaces

Surface evolution is more geometrically complex and computationally involved than for curves. But in $\mathbb{R}^3$, the codimension-one setting guarantees a unique unit normal at each point. This makes curvature-based motion both practical and flexible, enabling implicit representations such as level-set and phase field methods, and supporting a wide range of moving boundary problems with broad set of applications.

To describe the geometry of evolving surfaces, we introduce the following notation. Let $F : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a surface Σ at time t, with unit normal vector ν. The local shape is characterized by the principal curvatures κ_1 and κ_2, from which we define the mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and the Gaussian curvature $K = \kappa_1\kappa_2$. These quantities will form the basis for the flow laws considered below.

In this section we focus on extrinsic flows of surfaces in $\mathbb{R}^3$, which are governed by curvature-dependent normal velocity. We begin by reviewing several classical flows, each arising as the gradient flow of a natural geometric energy, and then explore their applications across physical and computational domains.

4.1. Examples of Surface Flows

Mean Curvature Flow: One of the most studied geometric flows, mean curvature flow evolves a surface by moving each point in the direction of the unit normal with speed equal to the mean curvature:

$$\partial_t F = -H\nu.$$

It is the L^2-gradient flow of surface area and plays a role analogous to curve shortening flow for curves: both are second-order, parabolic, and tend to smooth out geometric irregularities. However, the behavior of singularities and long-term evolution in the surface case is more intricate due to the richer topology and geometry of surfaces (Huisken, 1984). Critical points of the flow are minimal surfaces, which locally minimize area and appear in nature as soap films or bubbles spanning wire frames. For convex initial shapes, the flow contracts the surface to a round point in finite time. More generally, it is used in interface motion, grain boundary evolution, and surface fairing in geometry processing.

Willmore Flow: This flow seeks to minimize bending energy and governs the relaxation of thin elastic shells and biological membranes. It is given by $\partial_t F = -\Delta_g H - 2H(H^2 - K)$, where K is Gaussian curvature and Δ_g is the Laplace–Beltrami operator. It is the L^2-gradient flow of the Willmore energy

$$\mathcal{W}[F] = \int_\Sigma H^2 \,\mathrm{d}A - \int_\Sigma K \,\mathrm{d}A.$$

By the Gauss–Bonnet theorem, the integral of Gaussian curvature is $\int_\Sigma K \,\mathrm{d}A = 2\pi\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of the surface. The Willmore energy is invariant under conformal transformations.

Willmore flow is the natural surface analogue of elastic curve flow: its critical points, called *Willmore surfaces*, include spheres and Clifford tori. These structures appear in physical models of cell membranes (e.g., Canham–Helfrich theory), vesicles, and soft shells, where bending dominates the mechanical response.

4.2. Applications of Surface Flows

Surface flows play a crucial role in understanding and optimizing shape across physics, engineering, and biology. In materials science, mean curvature flow models the motion of grain boundaries in polycrystalline metals, affecting mechanical strength, corrosion resistance, and fatigue (Mullins, 1956). Related ideas appear in crystal growth, where the solidification of crystalline materials is modeled as a moving boundary problem in anisotropic environments, often requiring tools from Finsler geometry (Gurtin, 1993). Surface flows also describe natural wearing processes, such as erosion of stones or abrasion of materials, through curvature-driven evolution over long timescales (Firey, 1974).

In biology, the geometry of cell membranes is shaped by energetic considerations. The Canham–Helfrich energy, which combines mean and Gaussian curvature, leads to a modified Willmore flow that explains the morphology of red blood cells and vesicles (Brazda et al., 2020). These models are essential in understanding cellular mechanics, membrane elasticity, and morphogenesis.

Applications extend beyond the sciences into architecture and design, where minimal and Willmore surfaces provide efficient, aesthetically pleasing forms for tensile structures, domes, and bridges (Remešíková et al., 2014). In computer graphics and geometry processing, surface flows drive mesh fairing, shape interpolation, and smoothing. Tools like Brakke's Surface Evolver simulate curvature flows to optimize or animate surfaces with high visual or structural fidelity. The flexibility of surface flows, capturing complex deformation while respecting geometric principles, makes them indispensable across disciplines.

5. Conclusions and Outlook

Geometric flows provide a unifying language for modeling the evolution of shape, bridging abstract mathematics with real-world applications in physics, engineering, and beyond. From vortex filaments and dislocation lines to tensile structures and computational surfaces, these flows capture the geometry behind complex systems. As virtual and augmented reality, robotics, and interactive simulations grow in importance, the need for flexible, geometry-aware models becomes increasingly clear. We hope this brief exposition has inspired researchers, especially those in applied fields, to adopt geometric flows as part of their modeling toolkit, and to explore their potential as a natural, expressive, and rigorous framework for understanding and simulating the evolving shapes of the world.

Acknowledgments

This research was supported by the project *Modeling, prediction, and control of processes in nature, industry, and medicine powered by high performance computing* (Student Grant Agency of the Czech Technical University in Prague, No. SGS23/188/OHK4/3T/14), and by the project *Research Centre for Low-Carbon*

Energy Technologies (project of excellent research No. CZ.02.1.01/0.0/0.0/16_019/0000753), supported by the Operational Programme of Research, Development and Education, Ministry of Education, Youth and Sports of the Czech Republic.

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