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ELASTODYNAMICS IN PERIODIC MEDIA – APPROXIMATION BY HIGHER ORDER HOMOGENIZATION AND METAMATERIALS WITH RESONATORS

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Abstract: *We consider elastodynamics in periodically heterogeneous solids described by 1D continua. The homogenization based on the higher order asymptotic expansions is applied to derive effective (macroscopic) models. Relevance of these models is extended beyond the assumption of the perfect scale separation to respect finite size of the heterogeneities. These models involve higher order gradients enabling to interpret models of the generalized continua introduced using phenomenological approaches. Particular examples of bi- and triple-layered periodic composites are explored in the context of the wave dispersion analysis. It appears that a variety of models which approximate the response up to the 2nd order of accuracy with respect to the scale parameter can be used, leading to different dispersion properties. Due to the volume forces involved in the asymptotic analysis, structures with resonators can be represented to enhance band gap effects.*

Keywords: **Higher order homogenization, wave dispersion, elastodynamics, composite materials, metamaterials.**

1. Introduction

Wave propagation in periodically heterogeneous elastic media belongs to one of the most studied topics in the field of composite materials. However, a unified satisfactory description of this phenomena in the framework of the continuum mechanics remains cumbersome and ambiguities remains to interpret correctly different modelling approaches. Phenomenological extended continuum based theories involving higher order gradients provide micromechanically based models incorporating size effects and internal length scales, though not straightforwardly related to a specific microstructure under consideration. The homogenization based methods provide a promising alternative enabling to introduce internal length scales in a natural way. The asymptotic based homogenization of the 1st order provides the limit model of the Cauchy medium, where the microstructure size ℓ is infinitely small compared to a macroscopic size L, so that such models do not capture the dispersion properties emerging when wave lengths $L \not\gg \ell$. As the remedy, higher order homogenization leads to models with effective material coefficients computed directly for a given microstructure, whereby the internal length scale is retained. This issue has been discussed in several important works, namely comprising papers by Andrianov et al. (2008); Dontsov et al. (2013); Wautier and Guzina (2015); Cornaggia and Guzina (2020); Schwan et al. (2021). The present short paper contributes to these works by considering scale-dependent heterogeneity of the type reported in Rohan et al. (2009), such that the dispersion due to the finite scale ℓ/L interferes with the resonance features of the “metamaterial” with soft elastic components.

2. Elastodynamics in a periodic 1D continuum

We consider domain $\Omega =]0, L[\subset \mathbb{R}$ occupied by an elastic solid whose properties are given by an elasticity E^ε and the density ρ^ε, where $\varepsilon = \ell/L$ is the scale parameter, the ratio of the characteristic length featuring

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the micro- and macroscopic scales, respectively. The solid is represented by the 1D periodically heterogeneous continuum generated by copies of the (zoomed) representative periodic cell, $Y =]0, \bar{y}[$, such that $\Omega^\varepsilon = \bigcup_{k\in\mathbb{Z}}(\varepsilon Y + \varepsilon k\bar{y})$, where $\varepsilon\bar{y} = \ell$ is the real sized period length. The heterogeneity can be introduced via n subdomains $Y_i \subset Y$, such that $Y = Y_1 \cup \cdots \cup Y_n$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$.

Material parameters can be introduced due to the coordinated unfolding, $x = \xi + \varepsilon y$ with $y \in Y$ and $\xi = \varepsilon[x/\varepsilon]_Y$ being the "lattice coordinate".In this context, the material properties, *i.e.* the elasticity $E^\varepsilon(x) = E(y)$ and the density $\rho^\varepsilon(x) = \rho(y)$ are assumed to be piecewise continuous (even piecewise constant) in $Y_i \subset Y$, being functions $y = \{x/\varepsilon\}_Y = (x-\xi)/\varepsilon$. The unfolded spatial derivatives $\tilde{\partial}_x := \partial_x + \varepsilon^{-1}\partial_y$ of the displacements are used to derive the usual cascade of equations correspondingly to the employed asymptotic expansions.

Elastodynamic equation governing the displacement field u^ε is defined in any layer characterized by properties (E_i, ρ_i),

$$-\tilde{\partial}_x(E_i\tilde{\partial}_x u^\varepsilon) + \rho_i(\partial^2_{tt}u^\varepsilon - f^\varepsilon) = 0 \ , \quad \text{in any } \Omega \times Y_i \ , \tag{1}$$

where f^ε is a given generalized volume force.

3. Effective model with characteristic scales

3.1. Asymptotic analysis – higher order homogenization

In what follows, the time-dependence of displacement u^ε on t is considered, though t is not in the list of its arguments. The asymptotic expansion of the displacement is considered in its unfolded form,

$$u^\varepsilon(x) = \sum_{k=0,1,2,\ldots} \varepsilon^k u^k(x,y) \ , \quad u^k(x,y) = U^k(x) + \tilde{u}^k(x,y) \ , \quad U^k(x) = \left\langle u^k(x,\cdot)\right\rangle_Y \ , \tag{2}$$

where $\langle\ \rangle_Y$ is the average in Y. These expansions are substituted in (1), where the differential oprerator is unfolded

$$\begin{aligned} -\mathbb{L}^\varepsilon u^\varepsilon + \rho(\ddot{u}^\varepsilon - f^\varepsilon) = 0 \ , \quad x \in \Omega, \quad y \in Y_i \ , \\ \text{with } \mathbb{L}^\varepsilon(v) \equiv \mathbb{L}_{xx}(v) + \varepsilon^{-1}[\mathbb{L}_{xy}(v) + \mathbb{L}_{yx}(v)] + \varepsilon^{-2}\mathbb{L}_{yy}(v) \ , \end{aligned} \tag{3}$$

involving the differential operators $\mathbb{L}_{yy}\circ = \partial_y(E(y)\partial_y\circ)$, $\mathbb{L}_{xy} = \partial_x(E(y)\partial_y\circ)$, etc. Due to the linearity, for $k = 1,2,3$, $\tilde{u}^k$ can be expressed using the characteristic responses w^k, $k = 1,2,3$ (the so-called correctors of order ε^k), which are Y-periodic, *i.e.* $w^k(y) = w^k(y+\bar{y})$, satisfy in all subdomains Y_i, $i = 1,\ldots,n$ the following cascade of equations (complemented by interface conditions $[w^k] = 0$ and $[E\partial_y w^k + w^{k-1}] = 0$, where $[\,]$ is the jump on any interface between Y_i and Y_{i+1})

$$\begin{aligned} -\mathbb{L}_{yy}w^1 &= \partial_y E w^0 \ , w^0 \equiv 1 \ , \\ -\mathbb{L}_{yy}w^{k+1} &= \partial_y(Ew^k) + E(\partial_y w^k + w^{k-1}) - \rho w^{k-1}D^0/\rho_0 \ , \text{ for } k = 1,2 \ , \\ -\mathbb{L}_{yy}\varphi^3 &= \rho(M^1/\langle\rho\rangle_Y - w^1) \ , \end{aligned} \tag{4}$$

involving $\rho_0 = \langle\rho\rangle_Y$ and D^0, the standard effective elasticity, as defined below. Then the effective medium material properties can be computed

$$\begin{aligned} D^0 &= \left\langle E(1+\partial_y w^1)\right\rangle_Y \ , \\ D^2 &= \left\langle E(w^2 + \partial_y w^3)\right\rangle_Y \ , \\ M^k &= \left\langle \rho w^k\right\rangle_Y \ , \quad k = 1,2 \ , \\ M^3 &= \left\langle E\partial_y\varphi^3\right\rangle_Y \ . \end{aligned} \tag{5}$$

Based on (2), one can define the truncated averaged expansions of displacements $U^{(2)}(x,t) = U^0 + \varepsilon U^1 + \varepsilon^2 U^2$ and external forces $F^{(2)}(x,t) = F^0 + \varepsilon F^1 + \varepsilon^2 F^2$, such that the homogenized elastodynamic equation providing an approximation of (1) up to $o(\varepsilon^2)$ accuracy attains the following form

$$\mathcal{W}_{tx} \circ U^{(2)}(t,x) = \mathcal{F}_{tx} \circ F^{(2)}(t,x) \ , \tag{6}$$

where the operators $\mathcal{W}_{tx}$ and $\mathcal{F}_{tx}$, being parameterized by α and β are given,

$$\begin{aligned}\mathcal{W}_{tx} \circ U(t,x) &:= \left(1 + \varepsilon^2 r_2 (1-\alpha-\beta)\partial^2_{xx} + \varepsilon^2 r_2 \frac{\beta}{c_0^2}\partial^2_{tt}\right)\partial^2_{tt} U(t,x) \\ &\quad - c_0^2 \left[1 - \varepsilon^2\left(r_2\alpha - D^2/D^0\right)\partial^2_{xx}\right]\partial^2_{xx} U(t,x)\;, \\ \mathcal{F}_{tx} \circ F(t,x) &:= \left[1 - \varepsilon \frac{M^1}{\rho_0}\partial_x - \varepsilon^2 r_2 \left(\left(\alpha - \frac{\bar{M}^3}{M^2}\right)\partial^2_{xx} + \frac{\beta}{c_0^2}\partial^2_{tt}\right)\right] F(t,x)\;,\end{aligned} \tag{7}$$

involving the sound speed $c_0^2 = D^0/\rho_0$, and further coefficients $r_2 = M^2/\rho_0$, and $\bar{M}^3 = M^3 + \frac{(M^1)^2}{\langle\rho\rangle_Y}$. In, principle, parameters α, β can be chosen, whereby constraints are to be considered to ensure the hyperbolic character of (6), *cf.* Schwan et al. (2021). The external force $F^{(2)}$ can also be interpreted as the interaction force imposed by "resonators", as introduced below.

3.2. Periodic structures with resonators

The periodic structures like heterogeneous rods (rather than layered media) can be fitted with "ball-spring" couples which can be tuned to induce the acoustic band gaps. In the context of the asymptotic homogenization, these couples are characterized by the mass and the stiffness related proportionally to ε, see Rohan et al. (2009), which leads to the negative effective mass in the 1st order homogenized model, *i.e.* for $\varepsilon = 0$. Further we shall consider $M^1 = 0$ (which holds for any 2-component material), $\beta = 0$ and $\alpha = M^3/M^2$. For such a special case, denoting by $u_m^{(2)}$ the displacement of the resonator characterized by λ_m and Λ_m,

$$\begin{aligned}&\mathcal{W}_{tx} \circ U^{(2)} = \mathcal{F}_{tx} \circ F^{(2)}(t,x) = 0\;, \quad \text{with } F^{(2)} = \Lambda_m (u_m^{(2)} - U^{(2)})\;, \\ &\partial^2_{tt} u_m^{(2)} + \lambda_m (u_m^{(2)} - U^{(2)}) = 0\;,\end{aligned} \tag{8}$$

whereby also the operator $\mathcal{W}_{tx}$ is reduced due to $\beta = 0$, yielding the following equation,

$$\begin{aligned}&\Lambda_m \partial^2_{tt} U^{(2)} + (\lambda_m + \partial^2_{tt})\left(1 + \varepsilon^2 r_2 (1-\alpha)\partial^2_{xx}\right)\partial^2_{tt} U^{(2)} \\ &\quad - c_0^2 (\lambda_m + \partial^2_{tt})\left[1 - \varepsilon^2\left(r_2\alpha - D^2/D^0\right)\partial^2_{xx}\right]\partial^2_{xx} U^{(2)} = 0\;.\end{aligned} \tag{9}$$

3.3. Dispersion analysis

The influence of the higher order terms on the modelling of wave propagation can be studied using the classical dispersion analysis. For this, using the plane wave ansatz involving the wave number $\varkappa$ and the circular frequency ω is substituted in (6) which, in general, yields a bi-quadratic equation involving both $\varkappa^{2k}$ and ω^{2k}, $k = 1, 2$. The dispersion can be analyzed using the mappings $\varkappa^2 \mapsto \omega^2$, or $\omega^2 \mapsto \varkappa^2 =: \gamma$. For the latter alternative, the following quadratic equation is obtained for the model involving the resonators,

$$\begin{aligned}&A\gamma^2 + B\gamma + C = 0\;, \quad \text{where} \\ &\quad A = \varepsilon^2 c_0^2 (\alpha r_2 - D^2/D^0)(\lambda_m - \omega^2)\;, \\ &\quad B = (\lambda_m - \omega^2)[c_0^2 + \varepsilon^2\omega^2 r_2 (1-\alpha)]\;, \\ &\quad C = \omega^2[\omega^2 - (\Lambda_m + \lambda_m)]\;.\end{aligned} \tag{10}$$

Propagating wave modes exist for positive roots $\gamma = \varkappa^2 > 0$, while negative roots $\gamma = \varkappa^2$ indicate band gaps. When $\varepsilon = 0$, (10) reduces to one obtained for the 1st order homogenization result, $\omega = c_0\varkappa$. For the model without resonators, (10) still holds with $\Lambda_m = \lambda_m = 0$, which consequently enables to divide by ω^2. To illustrate the wave dispersion, we consider a simplified model extended only by $\partial^4_{xxxx} U$ w.r.t. the 1st order homogenized model, so that the microscopic length is involves in coefficient A through r_2 and D^2. In particular, for a bi-laminate structure represented by E_j and ρ_j in layers Y_j, $j = 1, 2$, we consider $\rho_1 = \rho_2$, whereas $E_2 \gg E_1$, and put $\alpha = 1$, $\beta = 0$, which yields (the case without resonators)

$$\omega^2 = K^\varepsilon(\varkappa^2) := \varkappa^2 c_0^2 \left(1 - \varepsilon^2 \frac{D^2}{D^0}\varkappa^2\right)\;, \tag{11}$$

which clearly yields two real frequencies for any $\varepsilon < (D^0/(D^2\bar{\varkappa}))^{1/4}$, noting that $\varkappa = \varepsilon\bar{\varkappa}$, where $\bar{\varkappa} = 2\pi/\bar{y}$ is expressed in terms of the period length $\bar{y}$. It is of interest to explore the influence of the resonators; in this case, (10) leads to

$$\omega^4 - \left(K^\varepsilon(\varkappa^2) + \Lambda_m + \lambda_m\right)\omega^2 - \lambda_m K^\varepsilon(\varkappa^2) = 0\;. \tag{12}$$

For $\varepsilon = 0$, (11) verifies the standard model, $\omega = c_0\varkappa = \sqrt{K^0(\varkappa)}$,however, due to the added resonators characterized by Λ_m and λ_m, the band gap effect appears for $\omega \in]\sqrt{\lambda_m}, \sqrt{\lambda_m + \Lambda_m}[$. For $\varepsilon > 0$, also the inner length $\ell^\varepsilon = \varepsilon\bar{y}$ is pronounced through the term $\varkappa^4$ in K^ε, see Fig. 1. Of the 2 roots of the bi-quadratic equation for $\gamma = \varkappa^2$, only one mode can propagate, if $\gamma > 0$, the stop bands are indicated for $\gamma < 0$.

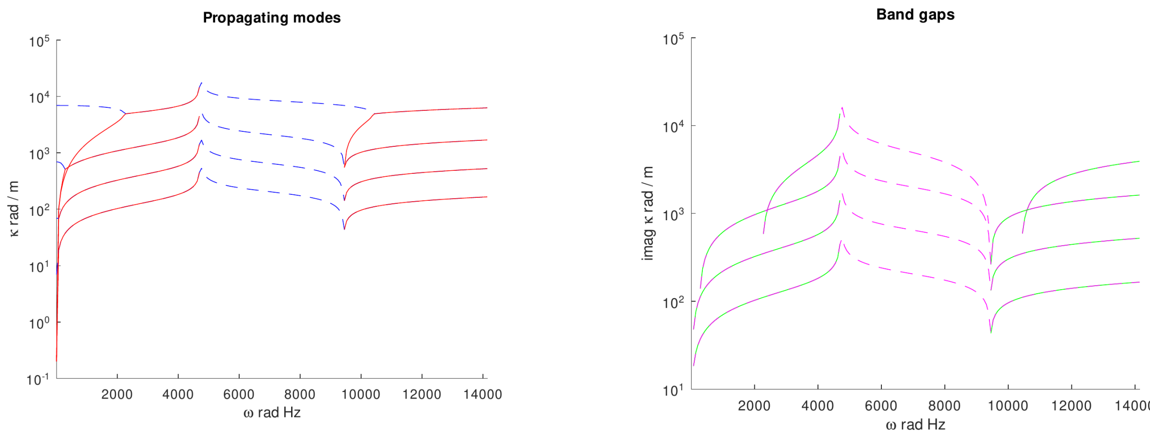

Fig. 1: Dispersion analysis $\omega \mapsto \varkappa$ for the model with resonators, Eq. (10), attaining the form (12). Clear band gaps between 4714 Hz and 9428 Hz. Four scales considered: $\varepsilon \in \{0.001, 01, 0.1, 1\}$.

4. Conclusion

The paper presents the higher order homogenization base modelling approach which provides the effective elastodynamics models involving higher orders temporal and spatial derivatives, namely $\partial^4_{xxxx}U$, $\partial^4_{tttt}U$ and $\partial^2_{xx}\partial^2_{tt}U$. Such models are comparable with the ones proposed on the purely phenomenological basis. Moreover, specific metamaterial features can be introduced, providing many further perspectives in the context of electro-mechanical devices with vast applications.

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