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PRACTICAL USABILITY OF FRACTIONAL VISCOELASTICITY FOR INTERLAYER POLYMERS

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Abstract: *Fractional calculus seems to be highly academic and artificial tool, but the opposite is true. The fractional calculus, i.e. the theory of derivatives and integrals of non-integer order, is applicable for describing many phenomena like signal processing or diffusion problems. For our research the prominent utilization is the fractional viscoelasticity for modeling a laminated glass interlayers. This theory can effectively describe polymer behavior in short term, where standard generalized Maxwell model is accurate only if utilizing a large number of cells. It seems that this approach is much natural, but on the other hand it brings a non-integer derivative of the displacement field into the governing equations, which induce some difficulties in numerical time integration. This paper is the beginning of research of the applicability of fractional models to describe the behavior of the interlayer in laminated glass.*

Keywords: Laminated glass, fractional viscoelasticity, generalized Maxwell chain, PVB interlayer, Fractional calculus.

1. Introduction

Fractional viscoelasticity, see (Koeller, 1984), assumes that stress is proportional to a non-integer time derivative of strain. This approach overcome some difficulties of standard models based on rheological combination of springs and dampers. The main drawback of such models is its impossibility to describe sufficiently the whole time domain. For example, in the generalized Maxwell chain model the precision is improved by adding more Maxwell cells, but extremely small and extremely large times are still not covered. The correct behavior is met only in a limit case, i.e., in a scheme with infinite number of Maxwell cells. The fractional viscoelasticity can be suitable tool for overcoming this difficulty and prospectively it can improve the accuracy of the description of viscoelastic materials with a smaller number of parameters. On the other hand this constitutive fractional relationship causes difficulties in performing time integration. For example, after spatial discretization by the finite element method, we obtain a set of fractional differential equations, which require the use of special integrators, see for example (Diethelm et al., 2005) or (Jacobs, 2020).

The application of fractional theory in laminated glass modeling was investigated for example by Paola et al. (2021), where the springpot was used as a rheological scheme. This approach leads to a linear relaxation function in the log-log space, which is adequate for a long time scale, but can be insufficient for high-velocity processes like impact, see the graph in Figure 2a. Therefore, more-element schemes and their practical usability are investigated in this article, based mainly on the results by Bonfanti et al. (2020).

2. Fractional viscoelasticity

The stress is proportional to the strain in elasticity whereas the stress is proportional to the strain rate in a viscous medium. The fractional viscoelasticity generalized this rule in the sense that all intermediate values

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are allowed, i.e. the stress σ is proportional to the α-derivative of the strain ε

$$\sigma(t) = \xi D^{\alpha}\varepsilon(t), \tag{1}$$

where D^{α} stands for a non-integer derivative operator of the order of α and ξ is proportionality constant. The limit case $\alpha \to 0$ represents an elastic spring whereas the case $\alpha \to 1$ represents a viscous damper. The parameter ξ has no clear physical meaning, so it is preferred to define this constant through elasticity E and viscosity η of limit cases as

$$\sigma(t) = E\tau^{\alpha}D^{\alpha}\varepsilon(t), \tag{2}$$

where $\tau = \eta/E$ is well-known relaxation time. This constitutive relationship is visually represented by rhombus and is called springpot, see Figure 1a. The springpot alone is one of the possible rheological models. As in classical viscoelasticity, we can combine the springpot with other rheological elements into more complex structures. Some useful equations are presented in following part and their applicability is investigated in the next section.

Springpot itself is described directly by equation (2), where setting $\varepsilon(t) = H(t)$ allows us obtain relaxation modulus $R(t)$. Here, the function $H(t)$ represents the Heaviside step function. The technicalities about fractional derivatives are not presented here because it is out of scope of this article. Recall, however, that at first glance, the trivial derivative on the right-hand side of equation (2) does not have a unique solution and we adopt the Riemann-Liouville fractional derivative rather than the Caputo one. The resulting relaxation function takes the following power form

$$R(t) = \frac{E}{\Gamma(1-\alpha)}\left(\frac{t}{\tau}\right)^{-\alpha}. \tag{3}$$

Maxwell cell is a serial connection of an elastic spring and a viscous damper. If the damper is replaced by the springpot, the fractional Maxwell model is obtained, see Figure 1b. The governing fractional differential equation

$$D^{\alpha}\sigma(t) + \frac{\sigma(t)}{\tau^{\alpha}} = ED^{\alpha}\varepsilon(t), \tag{4}$$

induces the following relaxation function

$$R(t) = E \cdot \mathrm{E}_{\alpha,1}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right), \tag{5}$$

where $\mathrm{E}_{\alpha,\beta}$ is the Mittag-Leffler function. It is worth noting the similarity of equation (4) with the ODE of classic Maxwell cell. The difference is that the first derivative was replaced by the α-derivative. Also the solution has the same structure, only the exponential function was replaced by the Mittag-Leffler function. In the limit case, it even holds that $\mathrm{E}_{1,1}(x) = \exp(x)$.

Generalized Maxwell model of N Maxwell cells and one spring can be further generalized by replacing all dampers with springpots, see Figure 1c. The solution of relaxation function is obtained superpositionally as

$$R(t) = E_{\infty} + \sum_{n=1}^{N} E_n \cdot \mathrm{E}_{\alpha_n,1}\left(-\left(\frac{t}{\tau_n}\right)^{\alpha_n}\right). \tag{6}$$

3. Applicability of models

This article is a part of the research focused on the laminated glass response, where certain class of polymers is used. From that reason, this section investigates the usability of fractional models for the Ethylene-vinyl acetate (EVA) and the Polyvinyl butyral (PVB) materials only, but the results can be qualitatively extended to other materials.

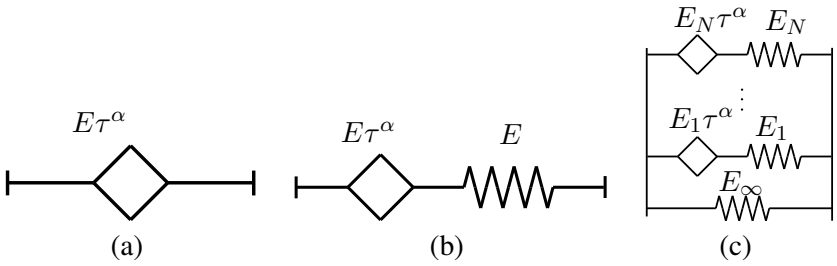

Fig. 1: Basic fractional rheological models: springpot (a), fractional Maxwell cell (b) and generalized fractional Maxwell model (c).

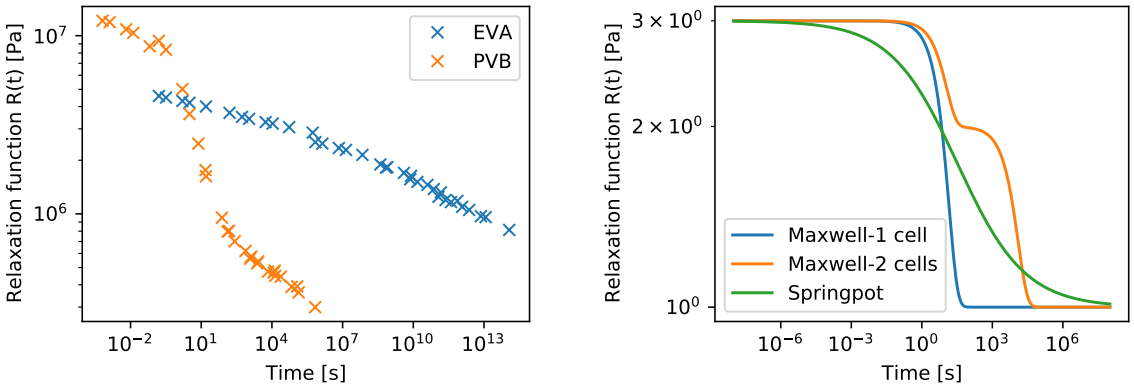

Fig. 2: Example of experimentally obtained relaxation modulus of EVA and PVB foil (a) and relaxation modulus induces by several models (b).

Although the springpot generalizes the elastic or viscous response, its applicability is not wide. It can be evident from the structure of the relaxation function (3), which is linear in the logarithm space. To the best of the author's knowledge, polymers in a laminated glass do not have this power-law property in the whole time domain and the springpot itself cannot be used. This is evident in Figure 2a, where experimentally obtained relaxation functions for PVB and EVA in time regime are plotted. The original data was obtained from plate-plate torsional rheometer, where a cylindrical specimen drilled out from a laminated glass was tested. The plotted data are the results of the utilization of time–temperature superposition principle to extend time domain. The details of experiments can be found in Hána et al. (2020).

Figure 2b displays the qualitative analysis of standard against fractional Maxwell model. The blue line is representative of the standard three-element Maxwell chain (spring and one Maxwell cell connected in parallel) with the relaxation time $\tau_1 = 1$ and $E_\infty = 1, E_1 = 2$. It is evident that this model can predict the viscoelastic behavior on the small time scale only, whereas at times farther from the relaxation time, the model predicts elastic behavior. This regime of interest can not be effectively enlarged and the same holds for the slope of the curve, which can not be change directly, but is determined indirectly from other parameters. Therefore, extra Maxwell cells with different relaxation times are added. As a result, larger time domain is described by the model, but with possible bumps, see the orange line in Figure 2b, where parameters $\tau_1 = 1, \tau_2 = 1000, E_\infty = 1, E_1 = 1, E_2 = 1$ were used. It is necessary to add another cells to smooth the curve. One way how to overcome this drawback is using the fractional Maxwell model, which has, through additional parameters α_n, the desired property - it can change the slope of the curve and better describe the behavior of the material, see the green line in Figure 2a, which represents model (6) with $\tau_1 = 1, E_\infty = 1, E_1 = 2, \alpha = 0.3$.

This advantageous behavior is also seen from the fitting process in Figure 3a and 3b, where the generalized standard and fractional Maxwell chain models are calibrated to experimental data from Figure 2a. The standard model can not sufficiently describe the observed behavior with a few cells and the only way to raise precision is to add more cells into the chain. Therefore 6 Maxwell cells, with relaxation times $\tau = \{10^{-2}, 10^{-1}, 10^0, 10^1, 10^2, 10^3\}$ for PVB and $\tau = \{10^1, 10^3, 10^5, 10^7, 10^9, 10^{11}\}$ for EVA, were used

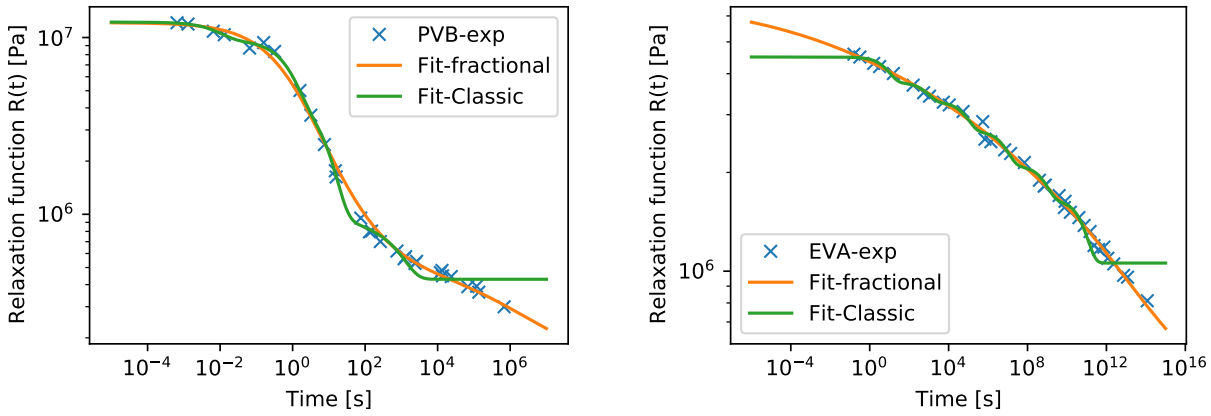

Fig. 3: Calibration of standard and fractional generalized Maxwell chain for experimentally obtained data of PVB (a) and EVA (b) foil.

to obtain a relatively precise response. In contrast, the fractional generalized Maxwell model can fit the data very accurately even when using the three-element fraction model, see orange line in Figure 3. Another benefit of the fractional model is its extensibility outside the experimentally measured time domain, which can be used not only to extrapolate the relaxation function, but also to better estimate the elastic modulus.

4. Conclusions

The applicability of fractional viscoelasticity for laminated-glass polymers was investigated in this article. This approach has both advantages and disadvantages. The main benefit of fractional viscoelasticity is natural description of the relaxation function using less number of parameters. However, it requires using a special numerical time integrator, which is not as effective as the standard one. Nevertheless, the use of fractional approach can be beneficial, especially for the description of the time domain outside the domain covered by the experiments.

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