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MODELLING OF ACOUSTIC STREAMING IN POROUS MEDIA

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Abstract: *The acoustic streaming of fluid in porous periodic structure is explored by Direct Numerical Simulations (DNS) and using the two-scale modelling based on the asymptotic homogenization. Using the classical perturbation approach, the first and the second order subproblems arising from the Navier-Stokes equations governing the barotropic viscous fluid dynamics in pores are obtained. The homogenization is employed to derive macroscopic models for both the subproblems. The driving force of the permanent flow is obtained due the time average of the nonlinear advection terms expressed using the first order acoustic harmonic fluctuations. The homogenization of the 1st order problem yields a wave equation involving the dynamic permeability. The DNS give illustrations of the phenomenon and provide a basis for validation of the homogenized model.*

Keywords: Acoustic streaming, Navier-Stokes equations, Homogenization, Porous media, Numerical simulations.

1. Introduction

The *acoustic streaming* (AS) appears due to inhomogeneities in viscous flow due to non-zero divergence of the Reynolds stress (due to the kinetic energy of the velocity fluctuations), or due to vibrating fluid-solid interface (effects of surface acoustic waves). It is observed at fluid boundary layers as the Rayleigh streaming due to the viscous phenomena (low frequencies), or in the bulk fluid as the high-frequency Eckart streaming. Major pioneering contributions are due to Nyborg (1953) and Lighthill (1978) who established the fundamental framework for the nonlinear acoustic wave treatment using the perturbation theory. AS in porous media has only rarely been reported in the literature (Raghavan, 2018). Arrays of cylinders (Valverde, 2015), or tubes (Panhuis et al., 2009) were considered while neglecting the solid phase compliance.

This paper presents the homogenization approach to the modelling of the AS of the Rayleigh type in rigid porous structures, though induced by a boundary effect. The derivation is based on the successive approximations (Nyborg, 1953) and on the asymptotic homogenization method, *e.g.* (Cioranescu et al., 2008). The fist method enables to linearize the Navier-Stokes equations in the context of the acoustic perturbations governed by two one-way coupled problems of the 1st and the 2nd order, whereas the latter provides macroscopic models for both these subproblems. Using a spectral decomposition of the so-called microflow problems, the effective dynamic permeability can be introduced. The AS velocity is given by the macroscopic pressure governed by the effective Darcy flow driven by the divergence of the Reynolds stresses. These are computed by a time averaging of the 1st order time-harmonic solution of the acoustic problem in the upscaled medium described by the dynamic Darcy law.

2. Mathematical model

The fluid flow of a barotropic viscous fluid in pores $\Omega_f \subset \Omega \subset \mathbb{R}^d$, $d = 2, 3$, of a rigid porous structure is described by the triplet $(\boldsymbol{v}^f, p^f, \rho^f)$ involving the fields of velocity, pressure and density, satisfying

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$$
\begin{aligned}
\rho^f \left(\frac{\partial}{\partial t} \boldsymbol{v}^f + \boldsymbol{v}^f \cdot \nabla \boldsymbol{v}^f \right) &= -\nabla p^f + \mu \nabla^2 \boldsymbol{v}^f + (\mu/3 + \eta) \nabla (\nabla \cdot \boldsymbol{v}^f) \;, \quad \text{in } \Omega_f \;, \\
\frac{\partial \rho^f}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}^f) &= 0 \;, \quad \text{in } \Omega_f \;, \\
p^f &= c_0^2 \rho^f + c_0 c_0' (\rho^f)^2 + o(|\rho^f|^3) \;, \quad c_0' = \partial_{\rho^f} c_0 \;,
\end{aligned}
\tag{1}
$$

where c_0 is the sound speed, μ, η are viscosity parameters. The boundary conditions can involve also vibrating surface of such a porous structure. For porosity $\phi = 1$, the waveguide is filled with the fluid only, see Fig. 1.

2.1. Successive approximations

Following the classical approach to the AS modelling, *e.g.* (Nyborg, 1953; Lighthill, 1978), we consider the following approximation of the flow response expressed by terms of different order with respect to a small parameter $\alpha \approx v_0/c_0$, where v_0 is a characteristic flow velocity, $v_0 \ll c_0$, such that

$$
\begin{aligned}
\boldsymbol{v}^f &= \alpha \boldsymbol{v}_1 + \alpha^2 \boldsymbol{v}_2 + \dots \;, \\
p^f &= p_0 + \alpha p_1 + \alpha^2 p_2 + \dots \;, \\
\rho^f &= \rho_0 + \alpha \rho_1 + \alpha^2 \rho_2 + \dots \;,
\end{aligned}
\tag{2}
$$

where p_0, ρ_0 are positive constants. By a_k we denote k-th order in α approximation of the quantity a. Moreover, we assume that any a_1 quantity is T-periodic in time, thus, $a_1 = \sin \omega t$ is a harmonic function given by the angular frequency $\omega = 2\pi/T$. Time averaging $\overline{(\quad)}$ over the time period of quantities $a_k = \boldsymbol{v}_1, p_1, \rho_1$ is applied below.

The 1st and 2nd order problems with respect to α are distinguished. At the first order, $o(\alpha^1)$, (1) yields

$$
\begin{aligned}
\frac{\partial}{\partial t} \rho_1 + \rho_0 \nabla \cdot \boldsymbol{v}_1 &= 0 \;, \\
\rho_0 \frac{\partial}{\partial t} \boldsymbol{v}_1 + \nabla p_1 &= \mu \nabla^2 \boldsymbol{v}_1 + (\mu/3 + \eta) \nabla (\nabla \cdot \boldsymbol{v}_1) \;, \\
p_1 &= c_0^2 \rho_1 \;,
\end{aligned}
\tag{3}
$$

at the second order, $o(\alpha^2)$, from (1) having applied the time averaging, we get

$$
\begin{aligned}
\frac{\partial}{\partial t} \bar{\rho}_2 + \rho_0 \nabla \cdot \bar{\boldsymbol{v}}_2 &= -\nabla \cdot \overline{(\rho_1 \boldsymbol{v}_1)} \;, \\
\rho_0 \frac{\partial}{\partial t} \bar{\boldsymbol{v}}_2 + \nabla \bar{p}_2 - \left(\mu \nabla^2 \bar{\boldsymbol{v}}_2 + (\mu/3 + \eta) \nabla (\nabla \cdot \bar{\boldsymbol{v}}_2) \right) &= -\rho_0 \left(\overline{(\boldsymbol{v}_1 \cdot \nabla) \boldsymbol{v}_1} + \overline{\boldsymbol{v}_1 (\nabla \cdot \boldsymbol{v}_1)} \right) \;, \\
\bar{p}_2 &= c_0^2 \bar{\rho}_2 + c_0 c_0' \overline{(\rho_1)^2} \;.
\end{aligned}
\tag{4}
$$

2.2. Homogenization

We consider periodic porous structures in domain Ω, such that the characteristic pore scale is proportional to a small parameter $\varepsilon = \ell/L$ defined by the ratio of the micro- and macroscopic characteristic lengths. The microstructure is determined by micropores $Y_f \subset Y$ in the representative periodic cell $Y =]0, 1[^d$, such that for a given ε, the pores Ω_f^ε are generated as a periodic lattice by the scaled representative pores εY_f, see Fig. 2. The scaling of the viscosity by ε is required correspondingly to the no-slip condition $\boldsymbol{v}^\varepsilon = \mathbf{0}$ on the pore walls Γ_{fs}^ε,

$$
\mu^\varepsilon = \varepsilon^2 \bar{\mu} \;, \quad \eta^\varepsilon = \varepsilon^2 \bar{\eta} \;. \tag{5}
$$

Other parameters describing the fluid are considered as constants, namely c_0, c_0', ρ_0, p_0. For the problem arising from (3), the unfolded solutions $(\boldsymbol{v}_1^\varepsilon, p_1^\varepsilon, \rho_1^\varepsilon)$ are expressed in terms of truncated expansions with respect to ε,

$$
\mathcal{T}_\varepsilon(\boldsymbol{v}^\varepsilon(x,t)) = \varepsilon^{1/2} \hat{\boldsymbol{v}}_1^0(x,y,t) \;, \quad \mathcal{T}_\varepsilon(p_1^\varepsilon(x,t)) = \varepsilon^{1/2} \left(p_1^0(x,t) + \varepsilon p_1^1(x,y,t) \right) \;, \tag{6}
$$

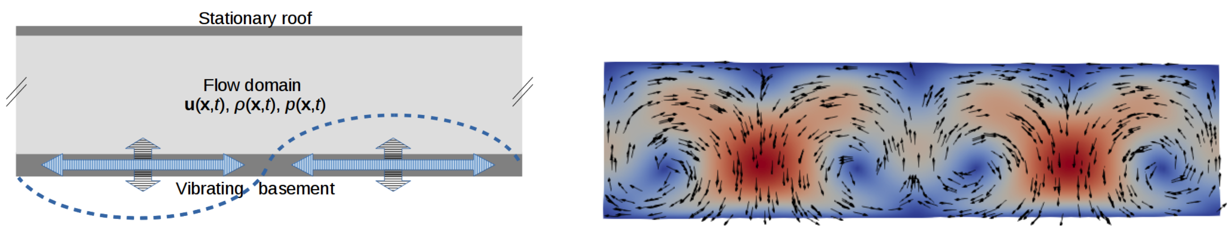

Fig. 1: Left: Macroscopic domain of a 2D porous waveguide with a vibrating basement (base $x_2 = 0$), the displacement: $\boldsymbol{u}(x,t) = \boldsymbol{U}\cos(2\pi x/L)\sin(2\pi t/T)$, where $\boldsymbol{U} = (U_1, U_2)^T$. Right: the acoustic streaming of a barotropic fluid for the boundary amplitude $\boldsymbol{U} = (U_1, 0)^T$.

where $y \in Y_f$ is associated with the macroscopic position $x \in \Omega$ by virtue of the unfolding operation, *i.e.* $y = (x - \hat{x})/\varepsilon$, where $\hat{x}$ is the center of a copy of the representative cell $\varepsilon Y(\hat{x})$, where $x \in \varepsilon Y(\hat{x})$. For the 2nd order time-averaged functions, the following truncated expansions are considered (for the sake of simplicity, we drop the bar $\overline{(\ )}$ from the notation of the limit functions, although these present the time averages; $\mathcal{T}_\varepsilon()$ is the periodic unfolding operator (Cioranescu et al., 2008))

$$\mathcal{T}_\varepsilon(\bar{\boldsymbol{v}}^\varepsilon(x,t)) = \boldsymbol{v}_2^0(x,y,t) + \varepsilon \boldsymbol{v}_2^1(x,y,t)\ , \quad \mathcal{T}_\varepsilon(\bar{p}_2^\varepsilon(x,t)) = p_2^0(x,t) + \varepsilon p_2^1(x,y,t)\ . \tag{7}$$

The homogenization $\varepsilon \to 0$ of the 1st order problem yields the dynamic Darcy law

$$\phi_f \frac{\partial p_1^0}{\partial t} - \nabla_x \cdot \int_0^t \boldsymbol{\mathcal{K}}(t-\tau)\nabla_x p_1^0(\tau,\cdot)\mathrm{d}\tau = 0\ , \qquad \text{with } \boldsymbol{\mathcal{K}}(t) = \rho_0^{-1}\sum_r \boldsymbol{\beta}^r \otimes \boldsymbol{\beta}^r \exp\{-\eta_r t\}\ , \tag{8}$$

where the permeability $\boldsymbol{\mathcal{K}}$ is obtained using an autonomous characteristic response of the microflow which is expressed in terms a spectral decomposition; this yields eigenpairs $\{\eta_r, \boldsymbol{w}^r\}$, with eigenvalues $\eta_r \in \mathbb{R}$ and Y-periodic eigenfunctions $\boldsymbol{w}^r(y)$ and related eigenvectors $\boldsymbol{\beta}^r = \int_{Y_f} \boldsymbol{w}^r \in \mathbb{R}^d$.

Further we consider a time-harmonic response, such as $p_1^0(t,x) = \hat{p}_1^0(x)\sin(\omega t)$. The acoustic streaming is governed by the 2nd order problem in which the driving force is constituted by the time averaged advection-related inertia terms $\overline{\boldsymbol{\mathcal{S}}}(\hat{\boldsymbol{v}}_1^0) = \overline{\hat{\boldsymbol{v}}_1^0 \cdot \nabla_y \hat{\boldsymbol{v}}_1^0}$, see $(6)_1$, being expressed as follows

$$\begin{aligned} \overline{\boldsymbol{\mathcal{S}}}(\hat{\boldsymbol{v}}_1^0) &= -\nabla \hat{p}_1^0 \otimes \nabla \hat{p}_1^0 : \sum_{r,s} \boldsymbol{\beta}^r \otimes \boldsymbol{\beta}^s \overline{\mathcal{M}_{rs}}(\omega)\boldsymbol{g}^{rs}\ , \\ \text{with } \boldsymbol{g}^{rs}(y) &= \boldsymbol{w}^r \cdot \nabla_y \boldsymbol{w}^s\ , \qquad \boldsymbol{\beta}^r = \int_{Y_f} \boldsymbol{w}^r\ , \\ \text{and } \overline{\mathcal{M}_{rs}}(\omega) &= \frac{\eta_r\eta_s + \omega^2}{2(\omega^2 + \eta_r^2)(\omega^2 + \eta_s^2)}\ . \end{aligned} \tag{9}$$

Homogenization of the 2nd order problem (4) leads to the non-stationary Darcy flow equation governing the time averaged pressure $\bar{p}_2^0(t,x)$,

$$\phi_f \frac{\partial \bar{p}_2^0}{\partial t} - \nabla_x \cdot \int_0^t \boldsymbol{\mathcal{K}}(t-\tau)\nabla_x \bar{p}_2^0(\tau,\cdot)\mathrm{d}\tau + \nabla_x \cdot \boldsymbol{W}^S(t,\cdot) = 0\ , \tag{10}$$

where $\boldsymbol{W}^S$ is the macroscopic acoustic streaming velocity, a linear functional of $\overline{\boldsymbol{\mathcal{S}}}(y)$, $y \in Y_f$, which is constant in time assuming the time harmonic response $\boldsymbol{v}_1^0(t,x)$.

3. Numerical simulations and comments

To illustrate the acoustic streaming in porous media, we consider a section of the 2D waveguide, as depicted in Fig. 1, Left, whose the lower wall vibrates, thus, generating an oscillatory velocity field in a boundary layer. In Fig. 1, Right, we display the time-averaged (non-oscillatory!) velocity field $\bar{\boldsymbol{v}}$, as computed by the DNS for the barotropic fluid filling the waveguide, thus, without any effect of the porosity (formally $\phi_f = 1$ in this case).

In Fig. 2, the acoustic streaming field $\bar{\boldsymbol{v}}$ is computed using DNS of the system (1) and porous 2D structures created by periodic arrays of circular obstacles. It can be conjectured that the macroscopic pattern is independent of the heterogeneity scale when the scale $\varepsilon \to 0$, which indicates that the acoustic streaming problem is homogenizable. The further research will focus on a validation of the two-scale homogenization-based model and its extension for deformable solid skeleton, *cf.* (Rohan and Naili, 2020; Rohan et al., 2021).

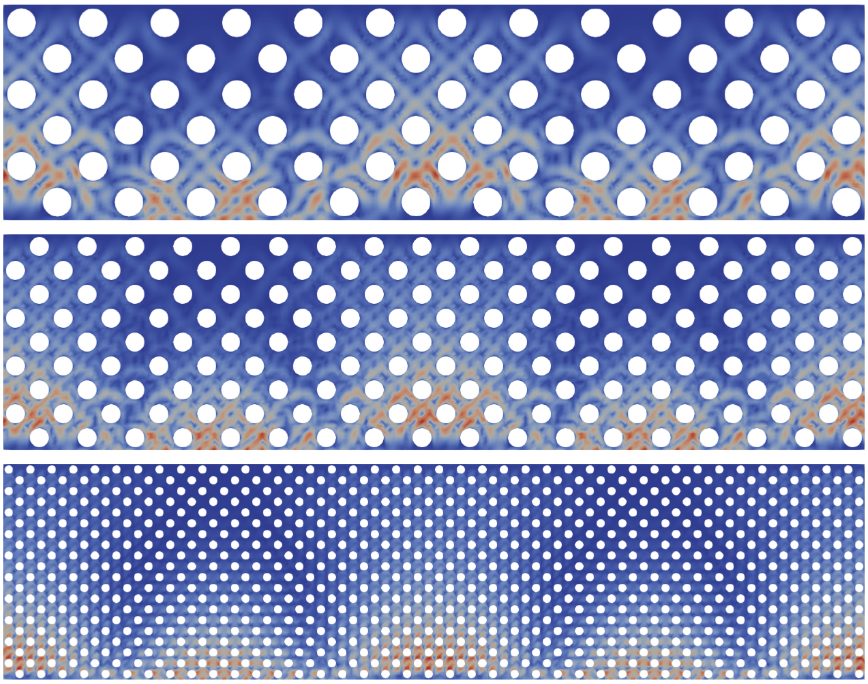

Fig. 2: Examples of the acoustic streaming for different scales ε of the porous structures in a 2D waveguide. The color means the $|\bar{\boldsymbol{v}}_2|$, computed by the DNS.

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