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THEORETICAL ANALYSIS OF SELECTED TRAJECTORIES INSCRIBED BY A BALL FREELY ROLLING IN A SPHERICAL CAVITY

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Abstract: *Indirect monitoring of structures is significantly complicated when an artificial vibration absorber has to be taken into account. Knowledge of the absorber's behaviour is necessary in order to correctly distinguish the response types obtained during an drive-by measurement. In this work, the mathematical model of the ball-type vibration absorber is used, which is based on the Lagrangian formalism. Three first integrals are identified when no external excitation nor damping is assumed. The paper illustrates the power of this approach, which enables a detailed analysis of free movement of the ball in the spherical cavity. Properties of several particular trajectories depending on initial conditions are presented.*

Keywords: Slender structure, Vibration absorber, First integral, Rolling sphere, Structural health monitoring.

1. Introduction

Contemporary slender and lightweight structures are naturally prone to vibration caused by natural ambient excitation or traffic induced forces. These structures are often designed to exhibit their slenderness and installation of the classical pendulum-based absorber is not possible for space, aesthetic, or other reasons. Various ball-type passive tuned mass absorbers then represent an alternative solution, because they are much less demanding in terms of a vertical space than conventional pendulum-type devices. This property makes such absorbers attractive especially for lightweight bridges.

On the other hand, the placement of such a damping device on the structure makes it difficult or impossible to indirectly measure the health of the bridge. These promising indirect measurement methods collect information on the condition of a structure from a passing vehicle or from a moving impulse load, see (Yang et al., 2004). The proposed drive-by procedure can be significantly complicated when the structure is equipped with an artificial damping device. The dynamic load acts along the entire driving path, the excitation intensity of the load is adjusted so that the response of the dominant natural mode is maximal. In contrast, the vibration absorbers are designed in such a way that this particular kind of response is maximally mitigated. An exact knowledge of the behaviour of the absorber is thus a necessary (although not sufficient) condition for correct interpretation of measured data. However, an effective procedure for such analysis is still the subject of research.

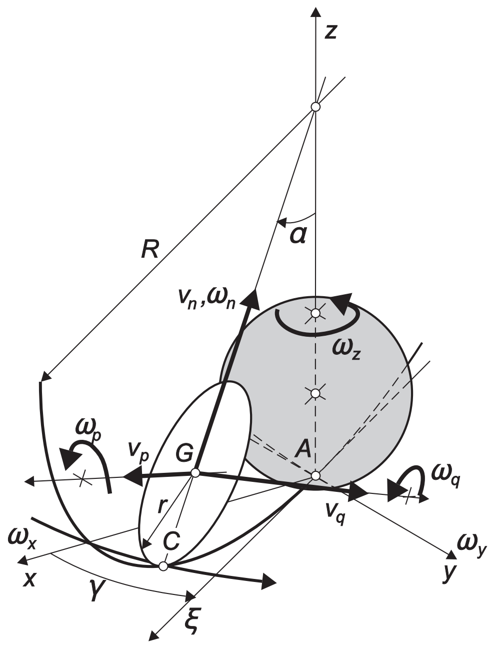

Fig. 1: Outline of the coordinate system

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Basic design guidelines of the ball-type absorber in a linearised 2D analogy and practical implementation experience was reported on by Pirner and Fischer (2000). A further analysis and stability assessment based on a non-linear 2D model was published later by the authors, (Náprstek et al., 2013). The complete governing non-linear differential system derived using the Appell-Gibbs approach was used by the authors for numerical identification of the auto-parametric resonance effects in (Náprstek and Fischer, 2020a) and also for the stability analysis (Náprstek and Fischer, 2020b). However, the conducted research was mostly based on numerical analysis. A novel mathematical model based on the Lagrangian approach is shortly introduced in this work, which follows the detailed analysis in (Náprstek and Fischer, 2021). When damping is neglected, three first integrals corresponding to the total energy of the ball can be formulated and exploited for identification particular states of a free response.

2. Mathematical model

The origin of moving coordinates is located in the center of the moving ball which moves along the concentric the sphere with radius $\varrho = R - r$. Moving axis p follows a tangent of the concentric sphere meridian in a vertical plane ξ, z, axis q is always horizontal and axis n follows a normal to the tangential plane in contact of both bodies being directed upwards, see Fig. 1. The angular velocities with respect to moving-coordinates axes p, q, n are denoted as $\boldsymbol{\omega} = (\omega_p, \omega_q, \omega_n)^T$ and are positive correspondingly with usual convention. Angles α, γ determine position of the ball center. Translational velocities of the ball center in moving coordinates are $\mathbf{v} = [v_p, v_q, v_n]^T$.

Basic formulae for kinetic and potential energies with respect to moving coordinates read:

$$T = \frac{1}{2}m\big[v_p^2 + v_q^2 + v_n^2 + \frac{2}{5}r^2(\omega_p^2 + \omega_q^2 + \omega_n^2)\big] \,, \tag{1a}$$
$$V = mg\varrho(1 - \cos\alpha) \,, \tag{1b}$$

where m, g represent the mass of the ball and the gravity acceleration, respectively. Because the spherical cavity has constant curvature $1/\varrho$, the projection of the moving coordinates into the cavity is represented by an orthogonal net. Thus, the relations between angles α, γ and velocities v_p, v_q read, see Fig. 1:

$$v_p \;=\; \varrho\dot{\alpha}, \quad v_q \;=\; \varrho\dot{\gamma}\sin\alpha, \quad v_n \;=\; 0. \tag{2}$$

The contact conditions of the perfect slipless rolling are given as:

$$r\omega_q - \varrho\dot{\alpha} = 0, \qquad r\omega_p + \varrho\dot{\gamma}\sin\alpha = 0. \tag{3}$$

Then the scaled total energy $E_0 = T + V$, see Eq. (1), can be formulated as follows:

$$\dot{\alpha}^2 + \dot{\gamma}^2\sin^2\alpha + \mu\omega_n^2 + 2\omega_0^2(1 - \cos\alpha) \;=\; E, \tag{4}$$

where

$$\text{where} \;\; \mu = \frac{2r^2}{7\varrho^2}, \quad \omega_0^2 = \frac{5g}{7\varrho}, \quad E = \frac{10E_0}{7m\varrho^2}.$$

Finally, three Lagrangian equations for three unknowns α, γ, ω_n are derived using the standard procedure and integrated to form the first integrals (invariants)

$$\ddot{\alpha} - (\dot{\gamma}^2\cos\alpha - \mu\omega_n\dot{\gamma} - \omega_0^2)\sin\alpha = 0 \,, \tag{5a}$$
$$\big(\dot{\gamma}\sin^2\alpha + \mu\omega_n\cos\alpha\big) = H \,, \qquad H = 5H_0/(7m\varrho^2) \,, \tag{5b}$$
$$(\omega_n) = S \,. \tag{5c}$$

Here S and H_0 represent the initial spin of the ball and the angular momentum of the system, respectively.

The behaviour of the sphere can be advantageously described by means of a "characteristic equation" which describes the instantaneous height δ of the sphere above the lower pole; $\delta = 1 - \cos\alpha$.

$$\dot{\delta}^2 = (E - \mu\omega_n^2 - 2\omega_0^2\delta)(2\delta - \delta^2) - (H - \mu\omega_n(1 - \delta))^2 = f(\delta) \,. \tag{6}$$

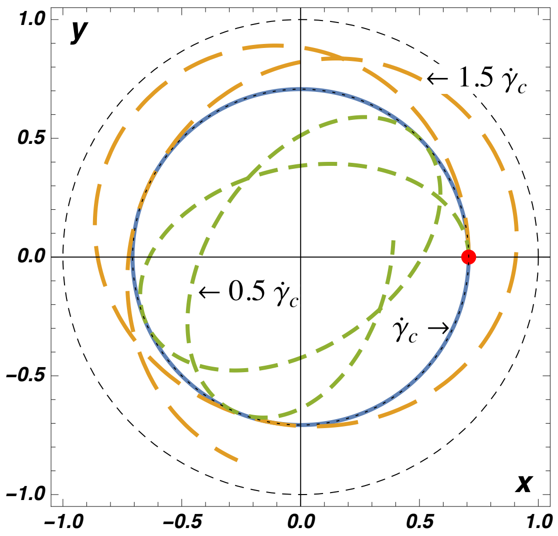

Fig. 2: Basic trajectories for $\omega_n = 0$ (no spin), $\dot{\gamma}_0 = \dot{\gamma}_c$: solid, $\dot{\gamma}_0 = 3/2\,\dot{\gamma}_c > \dot{\gamma}_c$: bright, $\dot{\gamma}_0 = 1/2\,\dot{\gamma}_c < \dot{\gamma}_c$: green

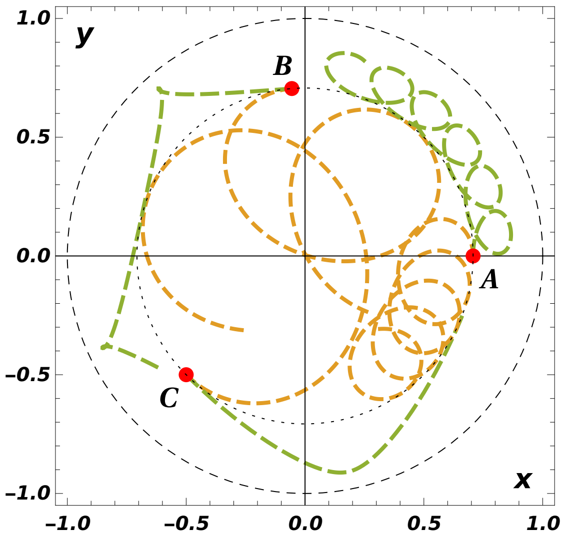

Fig. 3: Three trajectory types for $\omega_n \neq 0$. $\omega_n < 0$: green, $\omega_n > 0$: bright color. A: high value of spin, B: critical spin, C: low value of spin.

The cubic parabola $f(\delta)$ is physically meaningful only for values where $f(\delta) > 0$, as $\dot{\delta}$ should be real. Also, it can be shown that three roots $\delta_i, i = 1, 2, 3$ of $f(\delta)$ are such as

$$0 \leq \delta_1 \leq \delta_2 < 2 < \delta_3 \qquad \text{and} \qquad f(\delta) > 0 \text{ for } \delta_1 \leq \delta \leq \delta_2 \vee \delta \geq \delta_2. \tag{7}$$

Values $\delta_{1,2}$ represent a lower and upper limit of the trajectory of the sphere in the cavity. For geometrical reasons, only values $0 \leq \delta_{1,2} \leq 2$ can be considered. The zero and the coinciding roots can occur. They represent important cases and serve as certain limits between individual response types.

3. Selected response trajectories

Because no external excitation is assumed, particular trajectories are defined by an initial position of the ball and its initial velocity.

The circular trajectory, when the ball moves in a horizontal plane around the spherical cavity, is characterized by the condition $\delta_1 = \delta_2$. An easier derivation than the one based on Eq. (6) uses the assumption of constant deviation $\alpha(t) = \alpha_c$ and constant angular speed $\dot{\gamma}(t) = \dot{\gamma}_c$. Putting $\ddot{\alpha} = 0$ in Eq. (5a):

$$0 = \dot{\gamma}_c^2 \cos\alpha_c - \mu\dot{\gamma}_c\omega_n - \omega_0^2\,, \tag{8}$$

provides the relation for unknown initial speed $\dot{\gamma}_c$ in dependence on initial deviation α_c :

$$\dot{\gamma}_c = \frac{\mu\omega_n \pm \sqrt{4\omega_0^2\cos\alpha_c + \mu^2\omega_n^2}}{2\cos\alpha_c}\,. \tag{9}$$

If there is no initial spin of the ball, $\omega_n = 0$, initial speed $\dot{\gamma}_0 > \dot{\gamma}_c$ defines movement of the ball in waves above the original circular trajectory, Fig. 2, long dashes. Vice versa, for initial speed $\dot{\gamma}_0 < \dot{\gamma}_c$ the ball circulates below the limiting circle, see the green dashed line in Fig. 2.

Equation (8) enables to determine a correcting value of spin for an arbitrary initial speed $\dot{\gamma}_0$ (different from $\dot{\gamma}_c$) so that the resulting trajectory remains circular:

$$\omega_n = \frac{\dot{\gamma}_0^2\cos\alpha_c - \omega_0^2}{\mu\dot{\gamma}_0}\,. \tag{10}$$

A general influence of the spin velocity on the circular trajectory is shown in Fig. 3. The segments of trajectories originating in point A belong to a high value of negative (green, long dashed curve) and positive (bright, dashed curve) spin. The segments for low positive/negative values of spin begin in point C. The curves are distinctive by the absence of curls for a negative spin or by the loops circling the origin when the spin is positive. The limit between these two types is shown as segments originating in point B, which pass through the origin (positive) or possess sharp spikes (negative). For the negative spin is the limiting value identified by the existence of an infinite curvature of the trajectory at spikes. This condition gives rise to the following formula:

$$\omega_n = -\frac{7\dot{\gamma}_0 \omega_0^2 (R-r)^2 \sin^2\alpha_c}{r^2\left(\dot{\gamma}_0^2 \sin^2\alpha_c - 2\omega_0^2\cos\alpha_c\right)}\,. \tag{11}$$

For the positive spin, the limit trajectory passes through the origin and is characterised by the initial conditions:

$$\omega_n = \frac{7\dot{\gamma}_0 (R-r)^2 \left(\cos\alpha_c + 1\right)}{2r^2}\,. \tag{12}$$

Existence of curls is determined by vanishing speed $\dot{\gamma}$. This corresponds to points where the tangent to the trajectory directs towards the origin. The α coordinate of such points is given as follows:

$$\alpha_t = \cos^{-1}\left(\cos\alpha_c + \frac{7\dot{\gamma}_0 (R-r)^2 \sin^2\alpha_c}{2r^2\omega_n}\right)\,. \tag{13}$$

It is obvious that the argument of the inverse cosine in Eq. (13) has to fit its domain of definition. This condition, applied to ω_n, gives another way to the condition in Eq. (12).

4. Conclusions

The ball-type passive tuned mass vibration absorbers are popular damping devices which proved high efficiency in suppressing unwanted movement of slender structures exposed to wind excitation. Because its structural damping is low, the device is prone to a stability loss. For the same reason is the neglect of damping in the used mathematical model justifiable. Based on a fully 3D model, selected trajectories of a ball which freely moves in a spherical cavity were presented together with relations which specify the corresponding initial conditions. Appearance of such types of spatial behaviour in a real device may represent a danger for safety of the structure and, thus, adequate countermeasures has to be applied.

It also appears that shapes of complex trajectories of the ball within the cavity make a possible usage of indirect measuring techniques problematic. The analysis of the obtained data is significantly more complicated than it is in the case when the examined structures are not equipped with vibration absorbers. In addition, targeted excitation of the response in the first resonant mode may excite the absorber in an undesirable manner. In any case, solid knowledge of the behaviour of a possible absorber is a necessary step in developing an identification procedure.

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