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LOCALIZED FORMULATION OF BIPENALTY METHOD IN CONTACT-IMPACT PROBLEMS

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Abstract: *Often, the finite element method together with direct time integration is used for modelling of contact-impact problems of bodies. For direct time integration, the implicit or explicit time stepping are generally employed. It is well known that the time step size in explicit time integration is limited by the stability limit. Further, the trouble comes with the task of impact of bodies with different critical time step sizes for each body in contact. In this case, this numerical strategy based on explicit time stepping with the same time step size for both bodies is not effective and is not accurate due to the dispersion behaviour and spurious stress oscillations. For that reason, a numerical methodology, which allows independent time stepping for each body with its time step size, is needed to develop. In this paper, we introduce the localized variant of the bipenalty method in contact-impact problems with the governing equations derived based on the Hamilton's principle. The localized bipenalty method is applied into the impact problems of bars as an one-dimensional problem. The definition of localized gaps is presented and applied into the full concept of the localized bipenalty method.*

Keywords: Contact-impact problems, Explicit time integration, Bipenalty formulation, Localized Lagrange multipliers, Stability analysis.

1. Introduction

In finite element modelling of contact-impact problems, the penalty method is generally used, see (Wriggers, 2011). The classical penalty method for enforcing the impenetrability condition in contact mechanics produces the sensitivity of critical time step size on penalty stiffness parameter – with increasing the stiffness penalty parameter the critical time step size decreases. This behaviour is avoided by the bipenalty formulation, where the mass penalty term for enforcing of the gap rate for persistency condition is activated (González et al., 2021). By this way, the critical time step size is not chanced with a proper choice of the ratio of penalty parameters. Also, the stabilization of the spurious oscillations of contact forces can be realized with the predictor-corrector method, see (González et al., 2021; Kolman et al., 2021).

The partitioned analysis for impact problems based on localized Lagrange multipliers (Park et al., 2000) allows for total separation/splitting of governing equations of motion. Thus, the corresponding equations of motion with given boundary and initial conditions can be integrated with independent time step sizes. In this contribution, we suggest the combination of the bipenalty formulation of contact-impact problems together with the localized variant of Lagrange multipliers for correct solution of impact-contact problem. The governing equations for one-dimensional case is presented and discussed together with potential asynchronous time integration scheme.

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2. Strong formulation of contact initial-boundary value problem in 1D

The one-dimensional contact-impact problem for linear isotropic homogeneous bars is governed by the following constrained initial-boundary value problem (IBVP):

$$
\begin{cases}
E(x)u'' + b = \rho(x)\ddot{u} & \text{in } \boldsymbol{I} \times \boldsymbol{T} \\
u(x,0) = u_0(x) & \text{in } \bar{\boldsymbol{I}} \\
\dot{u}(x,0) = v_0(x) & \text{in } \bar{\boldsymbol{I}} \\
u(x,t) = \bar{u}(x) & \text{on } \boldsymbol{\Gamma}_u \\
Eu'(x,t) = \bar{\sigma}(x) & \text{on } \boldsymbol{\Gamma}_\sigma \\
g(t) \le 0 & \text{on } \boldsymbol{\Gamma}_\mathrm{c} \times \boldsymbol{T} \\
p_\mathrm{c} \ge 0 & \text{on } \boldsymbol{\Gamma}_\mathrm{c} \times \boldsymbol{T} \\
g(t)p_\mathrm{c} = 0 & \text{on } \boldsymbol{\Gamma}_\mathrm{c} \times \boldsymbol{T} \\
\dot{g}(t)p_\mathrm{c} = 0 & \text{on } \boldsymbol{\Gamma}_\mathrm{c} \times \boldsymbol{T}
\end{cases}
\tag{1}
$$

where $\boldsymbol{I} = \bigcup_i \boldsymbol{I}_i = (x_i^\ell, x_i^r)$; $i = 1, 2$ is the union of intervals of spatial points $x_i \in \boldsymbol{I}_i \subset \mathbb{R}$ defining the contacting bodies, and $\boldsymbol{T} = (0, t_\mathrm{end})$; $t_\mathrm{end} \in \mathbb{R}$ is the time interval. For one-bar contact problem with a rigid obstacle only one body is considered and $\boldsymbol{I} = \boldsymbol{I}_1$. In the first Eq. (1), which governs the balance of the linear momentum, $u(x,t)$ is the unknown displacement function, E is Young's elasticity modulus, and ρ is the mass density. b is the volume force per volume. Note that for the sake of simplicity, the second partial derivative with respect to x is denoted by double prime, $(\bullet)''$, whereas the second partial derivative with respect to t by superimposed dots, $(\ddot{\bullet})$. The governing equation $(1)_1$ is complemented with the initial conditions $(1)_{2,3}$, Dirichlet and Neumann boundary conditions $(1)_{4,5}$, and contact conditions $(1)_{6-8}$, where $g(t)$ is the gap function defined in the following section and $p_c(t)$ is the contact pressure. The last condition in $(1)_9$ is termed the persistency condition, see (Wriggers, 2011). In the initial conditions, $u_0(x)$ is the initial displacement function and $v_0(x)$ is the initial velocity function; both prescribed at time $t = 0$. Similarly for the boundary conditions, $\bar{u}(x)$ on Γ_u is the displacement function and $\bar{\sigma}(x)$ in Γ_σ is the traction function; both constant in time. Γ_u and Γ_σ are sets of boundary points where displacements and stresses are prescribed. Γ_b marks the boundary of domain of interest. Γ_c is the contact surface.

3. Definition of contact gap and localized gaps

At the first step, we have to define the gap function. The gap function $g(t)$ for the two-bar contact is defined as (Wriggers, 2011)

$$
g(t) = -\left[u(x_1^\ell, t) - u(x_2^\mathrm{r}, t) + g_0\right] = \mathbf{Z}^\mathrm{T}\mathbf{u} + g_0 \tag{2}
$$

where the geometrical meaning is depicted in Fig. 1.

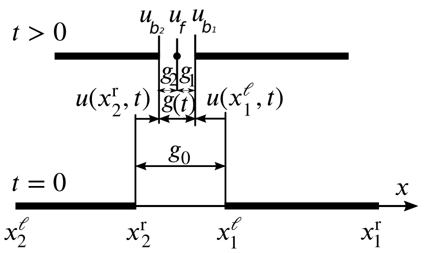

Fig. 1: Definition of contact gap and localized contact gaps for the problem of bars.

For the localized version of the bipenalty formulation we assume that the total gap is given by addition of the gaps from the boundary points of bars related to the contact frame and it is given by the frame displacement $u_f(t)$ as $g = g_1 + g_2$ where the localized gap function g_1 for body 1 is defined in 1D case as

$$
g_1(x_{b1}, x_f) = -[x_{b1} - x_f]n_1^f + g_{01} = -[X_{b1} + u_{b1} - (X_f + u_f)]n_1^f + g_{01}. \tag{3}
$$

In the same kinematical sense, we define gap function g_2 for body 2 as

$$g_2(x_{b2}, x_f) = -[x_f - x_{b2}]n_2^f + g_{02} = -[X_f + u_f - (X_{b2} + u_{b2})]n_2^f + g_{02} \tag{4}$$

where u_{b1} and u_{b2} are the displacements of the points of the boundaries of the body 1 a 2, resp., n_1^f is the normal vector at point x_f to the body 1 and n_2^f is the normal vector at point x_f to the body 2. Properties of the frame normal vector is as $n_2^f = -n_1^f$.

4. Weak form of contact-impact problem via localized bipenalty method - Hamilton's principle

We introduce the weak form of the contact-impact problems via the bipenalty method, for details see (González et al., 2021). The Lagrangian functional, $\mathcal{L}(u, \dot{u})$ of the problem of interest corresponding to Eq. (1) is given as

$$\mathcal{L}_{\mathrm{p}}\left(u, \dot{u}, u_b, \dot{u}_b, u_f, \dot{u}_f\right) = \mathcal{T}\left(\dot{u}\right) - \left(\mathcal{U}\left(u\right) - \mathcal{W}\left(u\right)\right) + \mathcal{W}_b\left(u, u_b, \lambda\right) + \mathcal{W}_c\left(u_b, \dot{u}_b, u_f, \dot{u}_f\right), \tag{5}$$

where

$$\mathcal{T}\left(\dot{u}\right) = \int_{\boldsymbol{I}} \frac{1}{2}\rho A \dot{u}^2 \,\mathrm{d}x, \quad \mathcal{U}\left(u\right) = \int_{\boldsymbol{I}} \frac{1}{2} E A u'^2 \,\mathrm{d}x, \quad \mathcal{W}\left(u\right) = \int_{\boldsymbol{I}} ubA \,\mathrm{d}x + \sum_{x\in\Gamma_\sigma} uA\bar{\sigma} \tag{6}$$

are the kinetic energy $\mathcal{T}$, the strain energy $\mathcal{U}$, and the work of external forces $\mathcal{W}$, respectively. Note, that the cross-section area of the bars is marked by A. The boundary term related to the method of Lagrange multipliers (Park et al., 2000) is given as

$$\mathcal{W}_{\mathrm{b}}(u, u_b, \lambda) = \int_{\Gamma_b} \lambda(u - u_b)\, d\Gamma \tag{7}$$

and the interface bipenalty term in the localized variant of bipenalty method related to the contact-impact problem yields

$$\mathcal{W}_{\mathrm{c}}\left(u_b, \dot{u}_b, u_f, \dot{u}_f\right) = -\int_{\Gamma_c} \frac{1}{2}\epsilon_{\mathrm{s1}} A \langle g_1\rangle^2 d\Gamma - \int_{\Gamma_c} \frac{1}{2}\epsilon_{\mathrm{s2}} A \langle g_2\rangle^2 d\Gamma + \int_{\Gamma_c} \frac{1}{2}\epsilon_{\mathrm{m1}} A \langle \dot{g}_1\rangle^2 d\Gamma + \int_{\Gamma_c} \frac{1}{2}\epsilon_{\mathrm{m2}} A \langle \dot{g}_2\rangle^2 d\Gamma, \tag{8}$$

where the operator $\langle\bullet\rangle$ are the so-called Macaulay's brackets defined as $\langle\bullet\rangle = \frac{|\bullet|+\bullet}{2}$ and ϵ_{s1}, ϵ_{s2}, ϵ_{m1} and ϵ_{m2} mark the localized stiffness and mass penalty parameters.

5. Explicit matrix form for the interface problem

The stationary solution of the weak form is then given in the matrix form for the 1D impact problems of two bars problem as follows

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_1^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b1}^T & \mathbf{0} & \mathbf{M}_{bb1}^P & \mathbf{0} & \mathbf{M}_{bf1}^P \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_{b2}^T & \mathbf{0} & \mathbf{M}_{bb2}^P & \mathbf{M}_{bf2}^P \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{bf1}^{PT} & \mathbf{M}_{bf2}^{PT} & \mathbf{M}_{ff}^P \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_1 \\ \ddot{\mathbf{u}}_2 \\ \lambda_1 \\ \lambda_2 \\ \ddot{\mathbf{u}}_{b1} \\ \ddot{\mathbf{u}}_{b2} \\ \ddot{\mathbf{u}}_f \end{bmatrix} +$$

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{bb1}^P & \mathbf{0} & \mathbf{K}_{bf1}^P \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{bb2}^P & \mathbf{K}_{bf2}^P \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{bf1}^{PT} & \mathbf{K}_{bf2}^{PT} & \mathbf{K}_{ff}^P \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \lambda_1 \\ \lambda_2 \\ \mathbf{u}_{b1} \\ \mathbf{u}_{b2} \\ \mathbf{u}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1^{ext} \\ \mathbf{f}_2^{ext} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_{0b1} \\ \mathbf{f}_{0b2} \\ \mathbf{f}_{0f} \end{bmatrix} \tag{9}$$

From (9), the accelerations of the frame is explicitly described in the form as

$$\ddot{\mathbf{u}}_f = (\mathbf{M}^P)^{-1}[\tilde{\mathbf{f}}_f - \mathbf{M}^{PT}_{bf1}(\hat{\mathbf{M}}^P_1)^{-1}\tilde{\mathbf{f}}_{b1} - \mathbf{M}^{PT}_{bf2}(\hat{\mathbf{M}}^P_2)^{-1}\tilde{\mathbf{f}}_{b2} - \mathbf{M}^{PT}_{bf1}(\hat{\mathbf{M}}^P_1)^{-1}\mathbf{L}^T_{b1}\left(\mathbf{B}^T_1\mathbf{M}^{-1}_1\mathbf{B}_1\right)^{-1}\mathbf{B}^T_1\ddot{\tilde{\mathbf{u}}}_1 - \mathbf{M}^{PT}_{bf2}(\hat{\mathbf{M}}^P_2)^{-1}\mathbf{L}^T_{b2}\left(\mathbf{B}^T_2\mathbf{M}^{-1}_2\mathbf{B}_2\right)^{-1}\mathbf{B}^T_2\ddot{\tilde{\mathbf{u}}}_2] \quad (10)$$

with the penalized mass matrix

$$\mathbf{M}^P = \mathbf{M}^P_{ff} - \mathbf{M}^{PT}_{bf1}(\hat{\mathbf{M}}^P_1)^{-1}\mathbf{M}^P_{bf1} - \mathbf{M}^{PT}_{bf2}(\hat{\mathbf{M}}^P_2)^{-1}\mathbf{M}^P_{bf2}. \quad (11)$$

From that, the boundary acceleration vector $\ddot{\mathbf{u}}_{b1}$ holds in explicit form

$$\ddot{\mathbf{u}}_{b1} = \left[\mathbf{L}^T_{b1}\left(\mathbf{B}^T_1\mathbf{M}^{-1}_1\mathbf{B}_1\right)^{-1}\mathbf{L}_{b1} + \mathbf{M}^P_{bb1}\right]^{-1}\left(\tilde{\mathbf{f}}_{b1} - \mathbf{M}^P_{bf1}\ddot{\mathbf{u}}_f + \mathbf{L}^T_{b1}\left(\mathbf{B}^T_1\mathbf{M}^{-1}_1\mathbf{B}_1\right)^{-1}\mathbf{B}^T_1\ddot{\tilde{\mathbf{u}}}_1\right) \quad (12)$$

Similarly, for the boundary acceleration vector $\ddot{\mathbf{u}}_{b2}$

$$\ddot{\mathbf{u}}_{b2} = \left[\mathbf{L}^T_{b2}\left(\mathbf{B}^T_2\mathbf{M}^{-1}_2\mathbf{B}_2\right)^{-1}\mathbf{L}_{b2} + \mathbf{M}^P_{bb2}\right]^{-1}\left(\tilde{\mathbf{f}}_{b2} - \mathbf{M}^P_{bf2}\ddot{\mathbf{u}}_f + \mathbf{L}^T_{b2}\left(\mathbf{B}^T_2\mathbf{M}^{-1}_2\mathbf{B}_2\right)^{-1}\mathbf{B}^T_2\ddot{\tilde{\mathbf{u}}}_2\right) \quad (13)$$

From that we can compute the Lagrange multipliers

$$\lambda_1 = \left(\mathbf{B}^T_1\mathbf{M}^{-1}_1\mathbf{B}_1\right)^{-1}\left(\mathbf{B}^T_1\ddot{\tilde{\mathbf{u}}}_1 - \mathbf{L}_{b1}\ddot{\mathbf{u}}_{b1}\right), \quad \lambda_2 = \left(\mathbf{B}^T_2\mathbf{M}^{-1}_2\mathbf{B}_2\right)^{-1}\left(\mathbf{B}^T_2\ddot{\tilde{\mathbf{u}}}_2 - \mathbf{L}_{b2}\ddot{\mathbf{u}}_{b2}\right) \quad (14)$$

and the acceleration of bodies can be computed as follows

$$\ddot{\mathbf{u}}_1 = \mathbf{M}^{-1}_1\left(\tilde{\mathbf{f}}_1 - \mathbf{B}_1\lambda_1\right) = \ddot{\tilde{\mathbf{u}}}_1 - \mathbf{M}^{-1}_1\mathbf{B}_1\lambda_1, \quad \ddot{\mathbf{u}}_2 = \mathbf{M}^{-1}_2\left(\tilde{\mathbf{f}}_2 - \mathbf{B}_2\lambda_2\right) = \ddot{\tilde{\mathbf{u}}}_2 - \mathbf{M}^{-1}_2\mathbf{B}_2\lambda_2 \quad (15)$$

with the predictions of the acceleration are $\ddot{\tilde{\mathbf{u}}}_1 = \mathbf{M}^{-1}_1\tilde{\mathbf{f}}_1, \quad \ddot{\tilde{\mathbf{u}}}_2 = \mathbf{M}^{-1}_2\tilde{\mathbf{f}}_2$ and the related forces then

$$\tilde{\mathbf{f}}_1 = \mathbf{f}^{ext}_1 - \mathbf{K}_1\mathbf{u}_1, \quad \tilde{\mathbf{f}}_2 = \mathbf{f}^{ext}_2 - \mathbf{K}_2\mathbf{u}_2, \quad \tilde{\mathbf{f}}_f = \mathbf{f}_{0f} - \mathbf{K}^{PT}_{bf1}\mathbf{u}_{b1} - \mathbf{K}^{PT}_{bf2}\mathbf{u}_{b2} - \mathbf{K}^P_{ff}\mathbf{u}_f \quad (16)$$

$$\tilde{\mathbf{f}}_{b1} = \mathbf{f}_{0b1} - \mathbf{K}^P_{bb1}\mathbf{u}_{b1} - \mathbf{K}^P_{bf1}\mathbf{u}_f, \quad \tilde{\mathbf{f}}_{b2} = \mathbf{f}_{0b2} - \mathbf{K}^P_{bb2}\mathbf{u}_{b2} - \mathbf{K}^P_{bf2}\mathbf{u}_f \quad (17)$$

where the matrices $\mathbf{K}^P_{xy}$ and $\mathbf{M}^P_{xy}$ have the meaning of the penalized stiffness and mass matrices.

6. Conclusions

The governing equations of the localized variant of the bipenalty method in contact-impact problems for the case of one-dimensional bar problems have been presented. Also, the explicit forms of the equations of motion of the interface based on the localized gaps have been found. These equations of motion together with kinematical equations give the full system for partitioned analysis with the asynchronous time stepping with time step sizes of bodies with non-integer ratios. The presented approach can help to accurate modelling of Split-Hopkinson pressure bar experiments.

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