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EFFICIENT PREDICTION OF THE PLASTIC COLLAPSE OF STRUCTURES

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Abstract: *The prediction of the collapse load (limit load) of a structure made of a material exhibiting elastic-plastic behavior is often of practical interest. The standard approach to obtain such collapse loads is based on iterative calculation schemes using classical nonlinear finite element methods. However, as an alternative approach the so-called finite-element-based limit analysis (FELA) can be applied. This approach is based on plastic limit theorems, first formulated by A. A. Gvozdev in 1936, and later independently by D. C. Drucker in 1952. Thereby, the collapse load is obtained as the minimum of a certain convex optimization problem, either considering kinematically compatible velocity fields (upper bound approach) or statically admissible stress fields (lower bound approach) within the structure, at the time instant of collapse. Thus, the whole load history does not need to be taken into account, resulting in a stable and numerically efficient approach compared to the standard scheme based on classical finite element formulations.*
The two significant disadvantages of the FELA method (the assumption of geometrical linearity and ideal plasticity) can be overcome by the so-called sequential finite-element-based limit analysis (SFELA). Thereby, the FELA method is called repeatedly, where the geometry and the plastic strain is updated after each iteration. A phenomenon which influences the collapse load of wooden structures is softening. Incorporation of softening in the framework of the SFELA has its limits however. Discussion of these limits and comparison with the so-called extended formulation of the FELA is provided. The capability of the FELA and SFELA methods is demonstrated on several numerical examples.

Keywords: Plastic collapse, Limit analysis, Sequential limit analysis, Second-order cone programming.

1. Introduction

The standard approach to obtain the collapse load (limit load) is based on iterative calculation schemes using the classical nonlinear finite element method (complete elasto-plastic analysis), as, e.g., can be found in (Robertson et al., 2005). However, alternative approaches can be applied: the finite-element-based limit analysis (FELA) or the elastic-reduction (also called the elastic-compensation) method, see, e.g. (Mellati et al., 2000). We focus on the former approach in this text.

2. Finite-element-based limit analysis method (FELA)

The FELA is based on limit theorems of plasticity, first formulated by A.A.Gvozdev in 1936. There are two basic assumptions:

1. Perfectly plastic assumption for the material
2. Small deformation assumption for the structure

There are two versions of the limit theorems (for simplicity we consider zero body forces) (Chen and Han, 1988):

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[1] The article appeared in a slim volume entitled *Proceedings of the Conference on Plastic Deformations, December* 1936, which was issued by the Academy of Sciences of the U.S.S.R. in 1938 under editorship of B.G.Galerkin (Gvozdev, 1938, 1960).

1. **The lower bound (static) theorem**: If an equilibrium distribution of stress $\boldsymbol{\sigma}$ can be found which satisfies boundary conditions, balances load $\boldsymbol{f}_\mathrm{e}$ applied on the loaded part of the boundary Γ and is everywhere below yield, ($f(\boldsymbol{\sigma}) < 1$), then the body at load $\boldsymbol{f}_\mathrm{e}$ will not collapse (Fig.1, left).

2. **The upper-bound (kinematic) theorem**: If a compatible mechanism $\dot{\boldsymbol{u}}$, and the corresponding plastic strain rate $\dot{\varepsilon}^\mathrm{p}$, is assumed, which satisfy the condition $\dot{\boldsymbol{u}} = 0$ on the displacement boundary Γ_u, then the load $\boldsymbol{f}_\mathrm{e}$, acting on the loaded part of the boundary Γ, determined from balance of power of external forces and internal dissipation power

$$\int_\Gamma \boldsymbol{f}_\mathrm{e}^\mathrm{T} \dot{\boldsymbol{u}} \, \mathrm{d}\Gamma = \int_\Omega \boldsymbol{\sigma} : \dot{\varepsilon}^\mathrm{p} \, \mathrm{d}\Omega \tag{1}$$

will be either higher than or equal to the actual limit load, i.e. the body will not stand up (Fig.1, right).

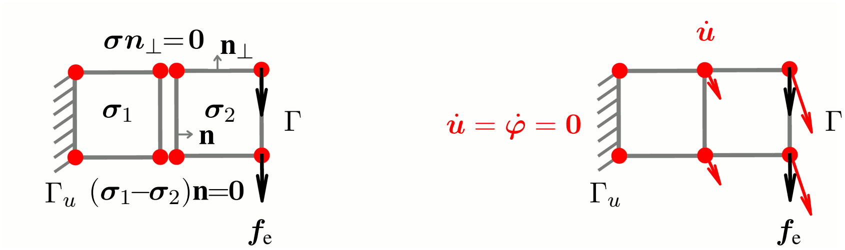

Fig. 1: Discretized structure. Left: lower bound formulation, right: upper bound formulation.

To determine the collapse load either of these two limit theorems can be used. In both cases the structure needs to be discretized into elements, in case of the lower bound formulation to equilibrium elements (Moitinho and Maunder, 2017), in case of the upper bound formulation to displacement-based elements (in which degrees of freedom are interpreted as velocities and not as displacements). Note that in the latter case, discontinuous elements are often used for numerical reasons (Ming et al., 2018). The optimization problem for the lower bound formulation takes the following form:

$$\max_{\boldsymbol{\sigma}} \lambda \quad \text{(load multiplier)} \tag{2}$$

$$\text{s.t.} \quad f(\boldsymbol{\sigma}) \leq 1, \quad \text{(yield criterion)} \tag{3}$$

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \quad \text{(equilibrium equation)} \tag{4}$$

$$(\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)\mathbf{n} = \mathbf{0}, \quad \text{(normal continuity between elements)} \tag{5}$$

$$\boldsymbol{\sigma}\mathbf{n} = \lambda \boldsymbol{f}_\mathrm{e}, \quad \text{(traction equals to the external load)} \tag{6}$$

where $\mathbf{n}$ is the normal vector and λ is the maximum load multiplier corresponding to the prescribed load $\boldsymbol{f}_\mathrm{e}$. The optimization problem for the upper bound formulation (in the case of shell structures) takes the following form:

$$\lambda = \min_{\dot{\boldsymbol{u}}, \dot{\boldsymbol{\varphi}}} d(\dot{\boldsymbol{\varepsilon}}^\mathrm{p}, \dot{\boldsymbol{\kappa}}^\mathrm{p}) \quad \text{(minimum of dissipation power)} \tag{7}$$

$$\text{s.t.} \quad d(\dot{\boldsymbol{\varepsilon}}^\mathrm{p}, \dot{\boldsymbol{\kappa}}^\mathrm{p}) = \max_{\boldsymbol{n}, \boldsymbol{m}, \text{ s.t. } f(\boldsymbol{n}, \boldsymbol{m}) \leq 1} [\boldsymbol{n}^\mathrm{T} \dot{\boldsymbol{\varepsilon}}^\mathrm{p} + \boldsymbol{m}^\mathrm{T} \dot{\boldsymbol{\kappa}}^\mathrm{p}] \tag{8}$$

$$\begin{bmatrix} \dot{\boldsymbol{\varepsilon}}^\mathrm{p} \\ \dot{\boldsymbol{\kappa}}^\mathrm{p} \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix}, \quad \text{(plastic strain rate approximation)} \tag{9}$$

$$\begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix}_{\Gamma_u} = \mathbf{0}, \quad \text{(kinematic boundary conditions)} \tag{10}$$

$$\boldsymbol{f}_\mathrm{e} \dot{\boldsymbol{u}} + \boldsymbol{m}_\mathrm{e} \dot{\boldsymbol{\varphi}} = 1, \quad \text{(power of external forces normed to one)} \tag{11}$$

where $\dot{\boldsymbol{u}}$ is the velocity, $\dot{\boldsymbol{\varphi}}$ is the rotational velocity, $\dot{\varepsilon}^\mathrm{p}$ is the in-plane plastic strain rate, $\dot{\boldsymbol{\kappa}}^\mathrm{p}$ is the bending plastic strain rate and $\boldsymbol{B}$ is the strain matrix. Note that the explicit form of the dissipation power $d(\dot{\varepsilon}^\mathrm{p}, \dot{\boldsymbol{\kappa}}^\mathrm{p})$ for the most practically used yield criteria is known. Both optimization problems (2-6), (7-11) are convex, i.e. uniquely and easily solvable, non-linear and in general non-smooth. Nonlinearity of these optimization problems is a consequence of the nonlinearity of the yield criterion. Historically, two solving strategies have been used, first using the nonlinear optimization algorithms, which turned out to be rather slow, and second using the linear programming algorithms, applied on the linearized yield function, which also turned out to be inefficient due to an enormous increase of the problem size. The game-changer came about with the primal-dual interior point method, which was found to efficiently solve the subclass of the nonlinear optimization problems called the second order cone programming (SOCP). This subclass includes only nonlinearities in the form of quadratic cones and covers most of the practically used yield criteria of the associated plasticity, among others, Tsai-Wu, Mohr-Coulomb and Drucker-Prager. A quadratic cone (=second order cone) is defined by

$$||\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}||_2 \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}+d,\ \ \boldsymbol{A}\in\mathbb{R}^{m\times n},\ \boldsymbol{x},\boldsymbol{c}\in\mathbb{R}^{n},\ \boldsymbol{b}\in\mathbb{R}^{m},\ d\in\mathbb{R}. \tag{12}$$

The primal-dual interior point method was proven to solve any SOCP problem to any accuracy in the polynomial time (Nesterov and Nemirovskii, 1994), which suddenly made the FELA approach in the late 20th century numerically very attractive. Among efficient numerical implementations of the primal-dual interior point method, which is also used in our simulations, is the optimization software MOSEK.

Due to the fact that the interior point method accepts non-smooth functions, multisurface plastic yield criteria can be considered without any further modification of the algorithm contrary to the standard incremental finite element approach, where Koiter's rule has to be carefully implemented.

There is an advantageous numerical property of the FELA – monotonicity of the iteration process if the mesh element size approaches zero. If the lower bound limit theorem is fulfilled pointwise, then it is guaranteed that the convergence is monotone, converging to the correct maximum load multiplier from below. Analogously, if the upper bound limit theorem is fulfilled pointwise, then it is guaranteed that the convergence is monotone, converging to the correct maximum load multiplier from above. If both lower and upper bound approaches are implemented, then the arithmetic mean of both values is a very good approximant of the correct maximum load multiplier on a given mesh. Note that the monotonicity of the convergence in case of standard finite elements is not guaranteed.

However, the FELA also has some principal disadvantages: neglects influence of material elasticity, assumes small deformations and assumes ideal-plastic material. Moreover, the FELA does not reflect the loss of stability due to elastic buckling which must be checked separately.

3. Sequential finite-element-based limit analysis method (SFELA)

The two basic assumptions of the FELA, i.e. assumption of small deformations and the assumption of ideal plasticity are both very restricting for real engineering applications. The sequential finite-element-based limit analysis (SFELA) is a modification of the FELA method, first introduced by Yang (1993), which overcomes, up to some extent, both restrictions. The idea of the method is based on the time integration of nodal velocities

$$\triangle\boldsymbol{u}_i = \triangle t\dot{\boldsymbol{u}}_i \tag{13}$$

where i is the node index and $\triangle t$ is the time step. This way we get a deformed mesh and the FELA can again be applied on this deformed mesh to get a new velocity field. This can be repeated on any time level j. Repeating this procedure results in

$$\triangle\boldsymbol{u}_i^j = \triangle t\dot{\boldsymbol{u}}_i^j. \tag{14}$$

The plastic strain rates can also be integrated in time to get increments of plastic strains

$$\triangle\boldsymbol{\varepsilon}_q^{\mathrm{p},j} = \triangle t\dot{\boldsymbol{\varepsilon}}_q^{\mathrm{p},j} \tag{15}$$

on time level j and in Gauss point q. The total plastic strains are given by

$$\boldsymbol{\varepsilon}_q^{\mathrm{p},j+1} = \sum_{k=1}^{j}\triangle\boldsymbol{\varepsilon}_q^{\mathrm{p},k}. \tag{16}$$

With this information we can, in the presence of hardening or even softening, go to the stress-strain diagram on the next time level $j+1$ and adapt the yield strength of each Gauss point accordingly. In this way, the effect of large deflections and hardening/softening is incorporated in the plastic collapse calculation.

4. Numerical examples

4.1. Example 1: Collapse of a wooden block with asymmetric holes

The collapse of a block with asymmetric holes shown in Fig.2, is calculated using the FELA method. The orthotropic material is described by the Tsai-Wu criterion:

$$\boldsymbol{\sigma}_{\text{loc}}^{\text{T}} \boldsymbol{A} \boldsymbol{\sigma}_{\text{loc}} + \boldsymbol{b} \boldsymbol{\sigma}_{\text{loc}} \leq 1,$$

$\boldsymbol{A} = \text{diag}[243, 65880, 65880, 11810, 29730, 11810]\ \text{GPa}^{-2};$

$\boldsymbol{b} = [-6.57, 85.6, 85.6, 0, 0, 0]\ \text{GPa}^{-1}.$

The 20-nodal hexahedron elements are used. The resulting distribution of the dissipation power for two different fiber angles is shown in Fig. 3.

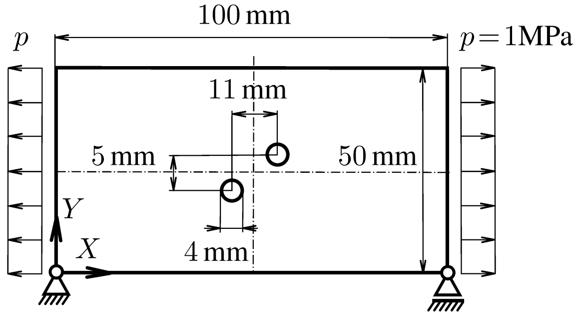

Fig. 2: Wooden block with asymmetric holes – problem sketch. Symmetric boundary conditions at $Z = \pm\frac{thickness}{2}$ are applied.

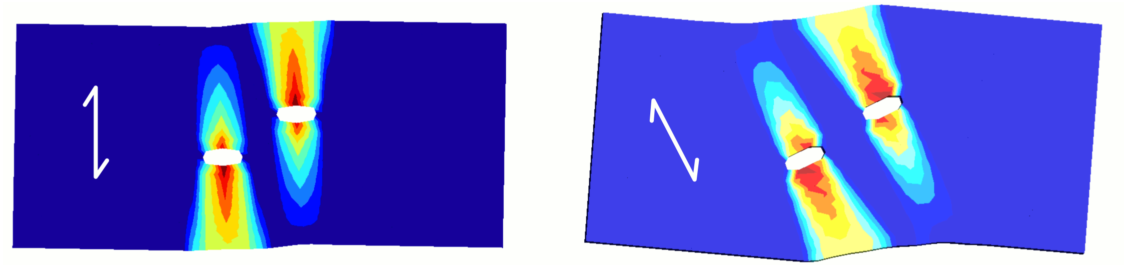

Fig. 3: Wooden block with asymmetric holes – dissipated power distribution depicted. Left: fiber direction: $(0, 1, 0)$, the maximum load multiplier $\lambda = 3.14$, right: fiber direction: $(-\sin 30°, \cos 30°, 0)$, $\lambda = 4.29$.

4.2. Example 2: Collapse of an S-rail

The collapse of an S-rail shown in Fig.4, is calculated by means of the SFELA method. Steel with isotropic hardening is considered: $\sigma_0 = A + B\,[\varepsilon^{\text{p}}]^n$, $A = 201.1\,\text{MPa}$, $B = 459.8\,\text{MPa}$, $n = 0.528$. The DKMQ24+ shell elements are used (Štembera and Füssl, 2020). The resulting distribution of logarithm of the dissipation power is shown in Fig. 5.

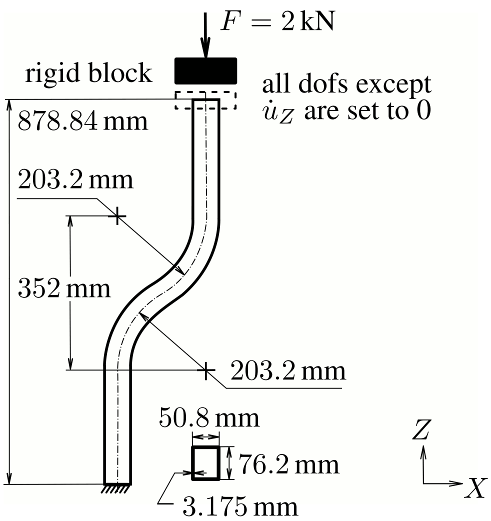

Fig. 4: S-rail – problem sketch.

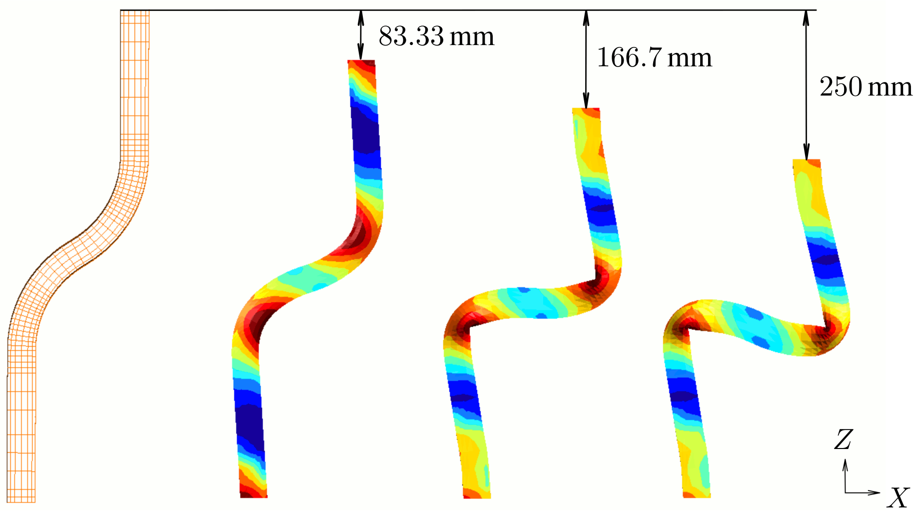

Fig. 5: S-rail – logarithm of the dissipated power depicted on the deformed structure in three consequent time steps. The maximum load multiplier equals to $\lambda = 21.4$.

5. Softening

When the collapse of wooden structures is calculated, you should be aware of the existence of softening in tension. The SFELA approach can include effects of softening on the post-collapse behaviour, it cannot however model a decrease of the collapse load due to softening (Fig.6). The reason is that the adaption of the yield strength due to the increase of the plastic strain is made at first at the second time level of the SFELA. Elasticity needs to be taken into account, which the SFELA cannot do, and can only be modelled by the extended limit analysis method (Manola and Koumousis, 2014). This method yields a non-convex optimization problem of the MPEC type (Mathematical Programming with Equilibrium Constraints) and takes typically the following form:

$$\max_{\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\Lambda}} \ \lambda \tag{17}$$

$$\text{s.t.} \quad \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\sigma} = \lambda \boldsymbol{f}, \tag{18}$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}^{\mathrm{e}}, \tag{19}$$

$$\varepsilon^{\mathrm{e}} = \boldsymbol{B}\boldsymbol{u} - \varepsilon^{\mathrm{p}}, \tag{20}$$

$$\varepsilon^{\mathrm{p}} = -\frac{\partial \Phi(\boldsymbol{\sigma}, \boldsymbol{\Lambda})}{\partial \boldsymbol{\sigma}} \boldsymbol{\Lambda}, \tag{21}$$

$$\Phi(\boldsymbol{\sigma}, \boldsymbol{\Lambda}) \geq \boldsymbol{0}, \tag{22}$$

$$\boldsymbol{\Lambda} \geq \boldsymbol{0}, \tag{23}$$

$$\Phi^{\mathrm{T}}(\boldsymbol{\sigma}, \boldsymbol{\Lambda})\boldsymbol{\Lambda} = \boldsymbol{0}. \quad \text{(complementarity condition)} \tag{24}$$

The global maximum must be numerically found with certainty, which is possible only by the means of the deterministic global optimization methods, for example by the branch and bound method. An efficient implementation is, e.g., the global optimizer BARON. However, this approach can handle a couple of hundred unknowns at most and is therefore unusable for real engineering applications.

6. Conclusions

The FELA method and its extension, the SFELA method, are outlined. The ability of the FELA method to determine the plastic collapse of the structure is demonstrated. The SFELA method includes effects of the large deformations and hardening in the calculation of the collapse and the post-collapse behaviour. The effects of the softening on the post-collapse behaviour is also taken into account, however the method is incapable of simulating the decrease of the collapse load due to softening, for which more sophisticated approaches are needed such as the extended limit analysis method.

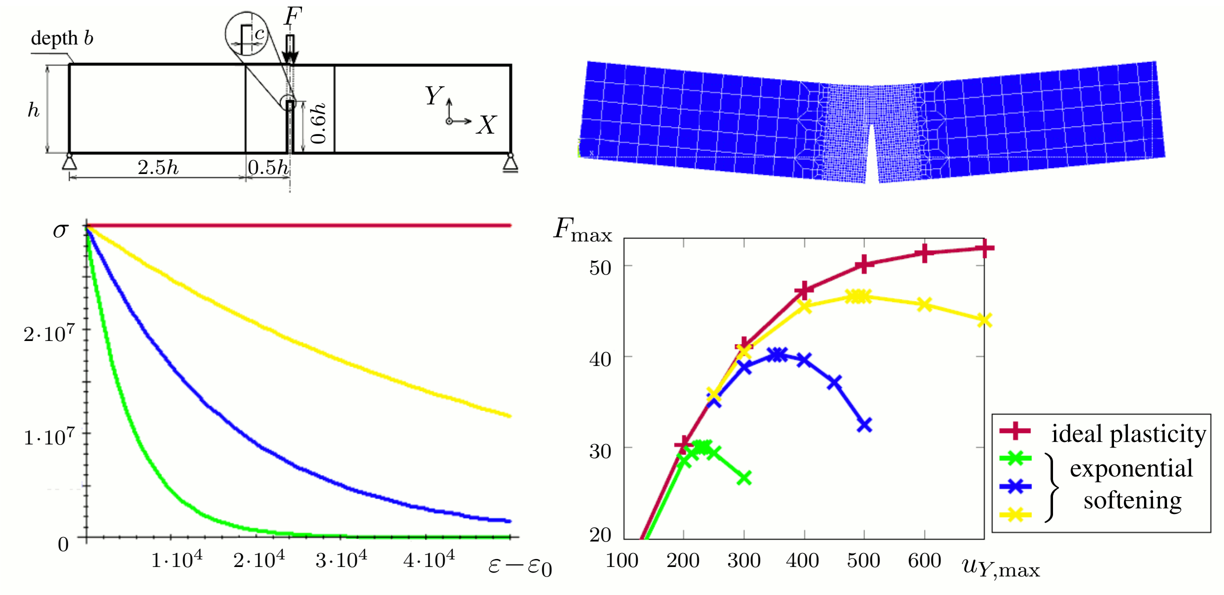

Fig. 6: A decrease of the collapse load due to an increase of the softening. This effect cannot be modelled using the SFELA method (modelled in ANSYS using the Voce plasticity model).

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