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MULTIFOLD STATIONARY SOLUTIONS OF AN AUTO-PARAMETRIC NON-LINEAR 2DOF SYSTEM

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Abstract: *A non-linear 2DOF model of a bridge girder with a bluff cross-section under wind loading is used to describe the heave and pitch self-excited motion. Existence conditions of stationary auto-parametric response for both the self-excited case and an assumption of a harmonic load form a non-linear algebraic system of equations. Number of distinct solutions to this algebraic system depends on the frequencies of two principal aero-elastic modes and other system parameters. Thus, the system may possess none, one, or several stationary solutions, whose stability has to be checked using the Routh-Hurwitz conditions. If all quantities entering the system are continuous functions, individual solutions may exhibit (piecewise) continuous dependence on selected system parameters. Thus, multiple identified solutions to the system for a given set of parameters may actually belong to a single solution branch and their values can be determined from the knowledge of the solution branch. Such a situation may significantly simplify assessment of stability of the particular solutions and/or provides an applicable overall description of the system response.*

Keywords: Multifold solution, Algebraic system, Aero-elastic system, Stationary vibration.

1. Introduction

Description of the behaviour of a dynamical system, which represents a structure, is an important part of its design. When the prospective structure is subjected to complex ambient excitation, the correct understanding of the response character is necessary. For example, vibration of a slender prismatic body in an air cross-flow results from the aero-elastic interaction between the non-conservative and gyroscopic forces and effects emerging due to vortex shedding processes. A sufficient description of the post-critical behaviour is of fundamental importance for functionality and safety of any structure.

The generally used single-degree-of-freedom (SDOF) or more complicated two-degree-of-freedom (2DOF) section models of a structure in an air stream represent a reasonable compromise between complexity and ability to characterise the dynamic processes, e.g., (Strømmen, 2006). Such simplified models are often successfully used in aerodynamic wind tunnel experiments (Král et al., 2014, Rizo et al., 2018). The 2DOF mathematical model used in this work includes a combination of the van der Pol and Duffing types of non-linearities. These are suitable for description of a strong quasi-periodic response, characterized by a beating effect which combines self-excited and forced vibrations. Subsequently, the presence of symmetric or asymmetric beating indicates an exchange of energy between individual degrees of freedom.

The non-linear model presented in this paper – a generalized van der Pol equation – introduces bi-quadratic damping terms into both coordinates in order to explain the concurrent effect of forced and self-excited vibrations which appears when the two vibration frequencies are close to each other and lead to a quasiperiodic response. The bi-quadratic terms may attain low or even negative values of damping which enables to model the auto-parametric resonance effects present in real structures. Detailed properties and effects of the bi-quadratic damping term in the van der Pol equation were studied by the authors in recent works, see (Náprstek and Fischer, 2019, Fischer and Náprstek, 2019) and references in these papers. The general properties of the 2DOF system used in this paper were published recently in (Náprstek and Fischer, 2020), where the authors paid a particular attention to differences between linear and non-linear models. The present work addresses particular details regarding identification of distinct stationary solutions.

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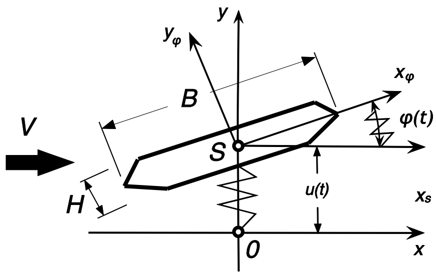

Fig. 1: Schematic 2DOF model of a bridge girder under wind loading; the response in heave u(t) (vertical direction) and pitch φ(t) (rotation around point S) is assumed.

The paper is organized as follows. After this introduction, Section 2 summarizes the non-linear mathematical model. Section 3 presents a procedure for identification stationary self-excited and harmonically forced solutions, their uniqueness is discussed in detail. Finally, Section 4 concludes.

2. Mathematical model

With respect to experimental results, two limit cycles (stable and unstable) determine behaviour of the non-linear response of a bridge girder, see a schematic plot in Fig. 1. Such behaviour of the theoretical model is made possible by the presence of fourth powers of variables u and φ in both damping terms. The non-linear mathematical model can be assumed as follows:

$$\begin{array}{lllll} \ddot{u} & +b_m(1-\nu_u u^2+\vartheta_u u^4)\dot{u} - hq\cdot\dot{\varphi} & +\omega_u^2\cdot u - p\cdot\varphi & = & Q_u(t) \\ \ddot{\varphi} & +q\cdot\dot{u}+b_I(1-\nu_\varphi\varphi^2+\vartheta_\varphi\varphi^4)\dot{\varphi} & +gp\cdot u+\omega_\varphi^2\cdot\varphi & = & Q_\varphi(t)\,. \end{array} \qquad (1)$$

Symbols $\omega_u^2,\ \omega_\varphi^2$ stand for the eigen-frequencies in heaving or pitching modes, respectively; b_m, b_I denote damping parameters; q is a gyroscopic coefficient; p represents non-conservative force; g, h are parameters that balance the physical dimensions in Eq. (1); e.g., $g = 1\ \mathrm{m}^{-2}, h = 1\ \mathrm{m}^2$. In the damping terms, symbols $\nu_u, \nu_\varphi\ [\mathrm{s}^{-1}\mathrm{m}^{-2}]$ characterize for positive values local destabilization in the neighbourhood of the trivial solution (or of the origin) due to increasing displacement amplitudes; $\vartheta_u, \vartheta_\varphi\ [\mathrm{s}^{-1}\mathrm{m}^{-4}]$ stabilize displacement amplitudes within a stable limit cycle, they are always positive. On the right hand side $Q_u, Q_\varphi\ [m\ \mathrm{s}^{-2}, \mathrm{rad\ s}^{-2}]$ represent an additive excitation.

The choice of the damping model in Eq. (1) causes instability of the trivial solution (i.e. zero response) and enables stabilization of the system at a certain stable limit cycle, even though such displacement amplitudes can become unacceptable from the viewpoint of system reliability. The model is suitable for description of the system in a resonant state (frequency locking). The excitation terms Q_u, Q_φ represent action of the vortex shedding. Both components are mutually connected by means of coefficients p, q.

Properties of possible stationary responses in both self-excited and harmonically excited states can be determined using the harmonic balance approach. For this purpose, the solution can be expected in the form

$$u \;=\; A_u\cos(\Omega t+\psi_u), \qquad \varphi \;=\; A_\varphi\cos(\Omega t+\psi_\varphi) \qquad (2)$$

where the response frequency Ω is close to the flutter frequency in the homogeneous case and to the excitation frequency when a harmonic excitation is assumed, see (Náprstek and Fischer, 2020) for details. In the former case, the stationary solutions (2) to the system (1) has to fulfil a non-linear algebraic system consisting from four equations for three response components $A_u, A_\varphi, \Delta\psi = \psi_u - \psi_\varphi$, and the flutter frequency Ω.

When a harmonic external excitation is assumed, $Q_u = \Phi_u \cos\Omega t$, $Q_\varphi = \Phi_\varphi \cos\Omega t$, the corresponding algebraic system for unknowns $A_u, A_\varphi, \psi_u, \psi_\varphi$ is given as follows:

$$\begin{aligned} 0 &= -b_m\Omega\left(1-\frac{1}{4}\nu_u A_u^2+\frac{1}{8}\vartheta_u A_u^4\right)A_u + (hq\Omega\cos\Delta\psi - p\sin\Delta\psi)\,A_\varphi - \Phi_u\sin\psi_u\,, \\ 0 &= -b_I\Omega\left(1-\frac{1}{4}\nu_\varphi A_\varphi^2+\frac{1}{8}\vartheta_\varphi A_\varphi^4\right)A_\varphi - (q\Omega\sin\Delta\psi + gp\cos\Delta\psi)\,A_u - \Phi_\varphi\sin\psi_\varphi\,, \\ 0 &= \left(\omega_u^2-\Omega^2\right)A_u - (hq\Omega\sin\Delta\psi + p\cos\Delta\psi)\,A_\varphi - \Phi_u\cos\psi_u\,, \\ 0 &= \left(\omega_\varphi^2-\Omega^2\right)A_\varphi - (q\Omega\sin\Delta\psi - gp\cos\Delta\psi)\,A_u - \Phi_\varphi\cos\psi_\varphi\,. \end{aligned} \qquad (3)$$

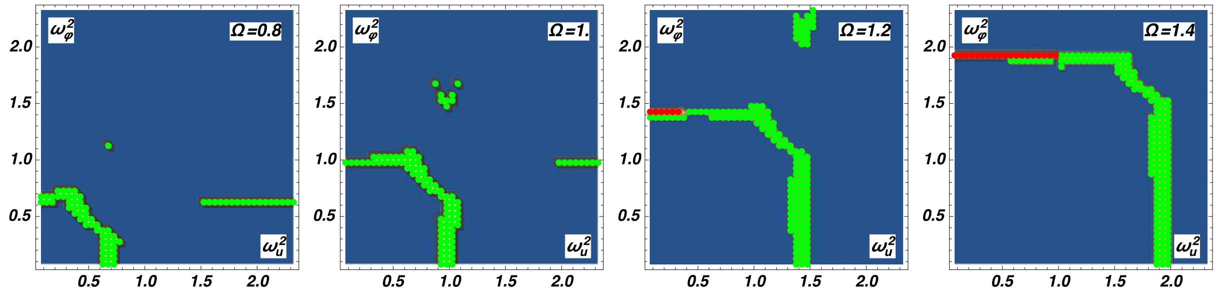

Fig. 2: Number of solutions in the $\omega_u^2 \times \omega_\varphi^2$ plane: blue – unique, green dots – threefold, red dots – fivefold solutions (excitation in the direction u only).

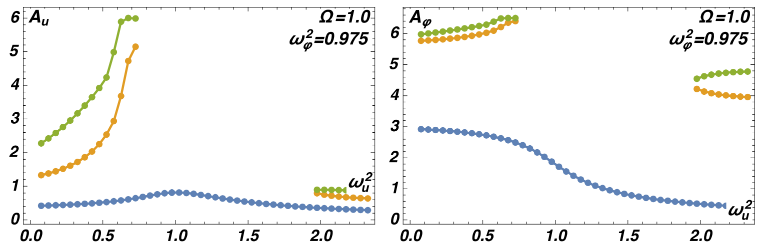

Fig. 3: Multiple solution branches for $\Omega = 1$ and $\omega_\varphi^2 = 0.975$ according to the numerical analysis.

3. Analysis

The non-linear systems Eq. (3) naturally possess none, one or more solutions for a given set of system parameters. Unfortunately, they have to be identified numerically because the explicit solution is not feasible. A numerical solution procedure identifies only single value, which may be, but does not need to be, close to an initial guess. To identify and analyse the major part of stationary solutions to Eq. (1), the system Eq. (3) was repeatedly solved for fixed system parameters $g = h = 1, p = q = 0.2,\ b_I = b_m = 0.2$ $\nu_u = \nu_\varphi = 0.5,\ \vartheta_u = \vartheta_\varphi = 0.025$ and the excitation amplitude $\Phi_u = 0.5, \Phi_\varphi = 0$; with initial values covering the complete area of interest $A_u, A_\varphi \in (0.05, 2)$, $\psi_u, \psi_\varphi \in (-\pi,\ \pi)$ (2 756 840 configurations) in the eigen-frequency and excitation-frequency range $\omega_u^2, \omega_\varphi^2 \in (0.01, 2.35)$ and $\Omega \in (0.1, 2)$ (total 83 942 frequency configurations). The found solutions were normalized ($0 < A_u, A_\varphi$ and $0 \le \psi_u, \psi_\varphi \le 2\pi$) to exclude repeated cases. In total, 85 618 distinct solutions were found for given frequency configurations, the vast majority was unique (91 %), only marginal number was threefold (8 %) and fivefold (1 %). A distribution of multifold solutions in the eigen-frequency plane is shown in Fig. 2. For an increasing excitation frequency the multifold solutions form a characteristic shape which corresponds to areas of increased stationary response, cf. (Náprstek and Fischer, 2020). The actual values of numerically identified amplitudes are shown in Fig. 3. The plots correspond to a particular configuration visible as horizontal section for $\omega_\varphi^2 = 0.975$ in Fig. 2, case $\Omega = 1$. The lowest blue curves depict the basic low amplitudes, which are present for all frequency configurations. Two brighter upper curves represent upper branches of possible stationary solutions.

The numerically obtained solutions provide only a quantitative information and their contribution to an overall insight is limited. Indeed, a numerical study brings a possibly huge set of discrete values, however, detailed information on their mutual connection is available only partially. On the other hand, the determined individual values of solutions can serve as good initial values for application of the numerical continuation technique, see, e.g., the monograph by Allgower and Georg (1990). This approach allows for determination of a general continuous dependence between individual parameters and enables to determine fine-grained details. An example is presented in Fig. 4. The plots show the numerically obtained amplitudes A_u, A_φ for fixed values of eigenfrequencies $\omega_u^2 = 0.375, \omega_\varphi^2 = 1.425$ depending on the excitation frequency (in the u component). The curves in Figs. 4a and 4c represent the resonance plots with a standard resonance peak around $\Omega \approx 0.6$ and the internal resonance zone around $\Omega \approx 1.2$, cf. the detailed plots in Figs. 4b and 4d. The numerical analysis revealed results indicated by the round dots. Mostly a only single blue dot is shown for each sample frequency, with an exception of $\Omega = 1.2$, where the numerical result

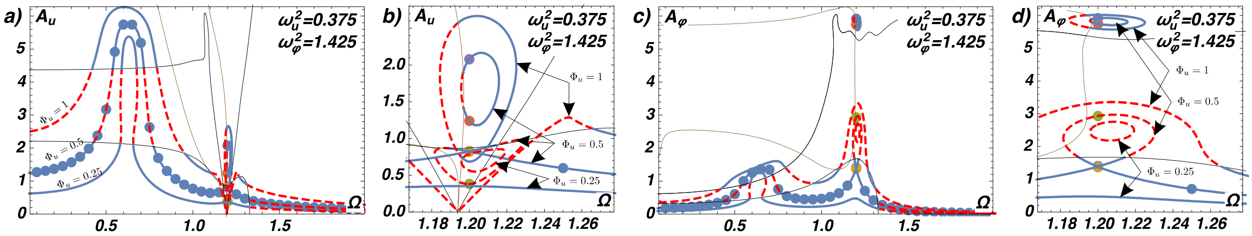

Fig. 4: Resonance curves of the 2DOF system for excitation amplitudes $\Phi_u = 0.25,\ 0.5,\ 1.0; \Phi_\varphi = 0$ computed numerically (dots) and using the continuation technique (lines); a),c): amplitudes A_u, A_φ, respectively, b),d): detailed plots for $\Omega \in (1.15, 1.27)$.

detected five different solutions indicated as dots with different colour, see details in Figs. 4b and 4d. The red and blue lines in this plot correspond to result obtained via numerical continuation with respect to the excitation frequency Ω, taking the five distinct values for $\Omega = 1.2$ as the starting points. It appeared that these five samples correspond to three disjoint resonance curves, one spanning the complete frequency interval and the other two form isolated closed curves. The curves in Fig. 4 are divided to blue solid and red dashed parts, which denote stable and unstable intervals according to the Routh-Hurwitz conditions. Thin black and brown lines denote stability limits corresponding to the individual RH conditions, their detailed description is, however, beyond scope of this work.

4. Conclusions

Numerical analysis has its indispensable position in a detailed investigation of complicated non-linear algebraic and differential systems. When properly used, it can provide results which are comparable to those obtained analytically. This was demonstrated on the bifurcation analysis of a 2DOF non-linear aeroelastic system. The presented methodology proved to be able to predict parameter areas where multifold stationary response can be expected.

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