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ON COMPUTING OF PULSE PROPAGATION AND REFLECTION IN 2D ELASTIC WAVEGUIDE

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Abstract: *Pulse propagation in elastic waveguides is simulated by means of finite volume methods. Results of calculations by means of the standard wave-propagation algorithm are compared with those obtained by the thermodynamically consistent excess quantities method. The main difference in these approaches is in the implementation of boundary conditions. The similarity and the distinction of the results are demonstrated.*

Keywords: Elastic waveguide, Pulse propagation, Finite-volume method, Boundary conditions

1. Introduction

A complex behavior of waves in elastic waveguides is due to the interaction of a wave with lateral boundaries. Propagation of a pulse demonstrates even more complicated motion. This motion can be described in terms of hyperbolic conservation laws. A lot of numerical methods were applied to their solution: finite-difference methods (Godlewski and Raviart, 1996; Trangenstein, 2009), finite-element methods (Cohen, 2002; Kampanis et al., 2008), discontinuous Galerkin methods (Hesthaven and Warburton, 2007; Cohen and Pernet, 2017), finite-volume methods (LeVeque, 2002; Guinot, 2003), spectral methods (Hesthaven et al., 2007; Gopalakrishnan et al., 2007) etc. The comprehensive survey of numerical methods for conservation laws is presented recently (Hesthaven, 2018).

Finite volume schemes are powerful numerical methods for solving nonlinear conservation laws and related equations. Such methods are locally conservative and based on cell averages. The numerical solution of systems of hyperbolic conservation laws is dominated by Riemann-solver-based schemes (LeVeque, 2002; Guinot, 2003). The upgrade of the solution in a given cell is determined by the exchanges (via fluxes) at the interfaces with the neighbouring cells. However, the cell average of a solution in a cell contains too little information. In order to obtain higher-order accuracy, neighboring cell averages are used to reconstruct an approximate polynomial solution in each cell. This reconstruction procedure is the key step for many high-resolution schemes (Liu et al., 2007).

When extending the flux-difference schemes to multi-dimensional problems, the so-called grid aligned finite volume approach or dimensional splitting method is adopted traditionally using one-dimensional Riemann solvers. However, for multi-dimensional problem, there is in general no longer a finite number of directions of information propagation. It has been pointed out (Roe, 1986) that the Riemann-solver is applied in the grid- rather than the flow-direction, which may lead to a misinterpretation of the local wave structure of the solution. To overcome the drawbacks of methods based on dimensional splitting, there have been considerable efforts to develop so-called genuinely multi-dimensional schemes for solving hyperbolic conservation laws (Colella, 1990; Billet and Toro, 1997; LeVeque, 2002; Guinot, 2003).

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2. Governing equations

Elastic solids are characterized by the Hooke law which can be represented in the isotropic case in the form of the stress-strain relation (Mase et al., 2009)

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \tag{1}$$

with the Cauchy stress tensor σ_{ij}, the strain tensor ε_{ij}, and the Lamé parameters λ and μ. In the linear elasticity, a motion is governed by the local balance of linear momentum at each regular material point (Achenbach, 1973)

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \tag{2}$$

where ρ is the matter density, v_i is the particle velocity, t is time, f_i is a body force, and x_i are spatial coordinates. In the plane strain case in the absence of body force, the governing equations for wave motion (2) are reduced to

$$\rho \frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y}, \tag{3}$$

$$\rho \frac{\partial v_2}{\partial t} = \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y}. \tag{4}$$

Stress-strain relations (1) are reformulated accordingly

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda \varepsilon_{22}, \tag{5}$$

$$\sigma_{12} = \sigma_{21} = 2\mu \varepsilon_{12}, \tag{6}$$

$$\sigma_{22} = (\lambda + 2\mu)\varepsilon_{22} + \lambda \varepsilon_{11}. \tag{7}$$

Time derivatives of stress-strain relations (5) – (7) represented in terms of velocities together with the balance of linear momentum (3)–(4) form the closed system of equations, which is convenient for a numerical solution.

3. Averaged and excess quantities

Let us introduce a Cartesian grid of cells $C_{nm} = [x_n, x_{n+1}] \times [y_m, y_{m+1}]$ with interfaces $x_n = n\Delta x, y_m = m\Delta y$, and time levels $t_k = k\Delta t$. For simplicity, the grid size $\Delta x, \Delta y$ and time step Δt are assumed to be constant. The values of wanted fields are somehow distributed across the cells.

The main idea in the construction of the algorithm is the consideration of every computational cell as a thermodynamic system (Muschik and Berezovski, 2004). Since we cannot expect that such thermodynamic system is in equilibrium, its local equilibrium state is described by averaged values of field quantities. The use of cell averages is the standard procedure in the finite-volume methods. What is non-standard that is the introduction into consideration so-called "excess quantities" in the spirit of the thermodynamics of discrete systems (Muschik and Berezovski, 2004). The excess quantities represent the difference between values of true and averaged quantities (Berezovski et al., 2008; Berezovski, 2011):

$$v_i = \overline{v}_i + V_i, \quad \sigma_{ij} = \overline{\sigma}_{ij} + \Sigma_{ij}. \tag{8}$$

Here overbars denote averaged quantities and capital letters relate to excess quantities.

3.1. Numerical scheme

Keeping in mind the representation of field quantities mentioned above, we integrate the governing equations over the computational cell. The result of the integration is expressed in terms of excess quantities at the boundaries of the cell. The numerical scheme follows from the standard approximation of time derivatives

$$\dot{f} \approx \frac{f^{k+1} - f^k}{\Delta t} \quad \forall f, \tag{9}$$

and the definition of averaged values for velocities and stresses

$$\overline{v}_i = \frac{1}{\Delta x \Delta y} \int_{\Delta x} \int_{\Delta y} v_i dx dy, \quad \overline{\sigma}_{ij} = \frac{1}{\Delta x \Delta y} \int_{\Delta x} \int_{\Delta y} \sigma_{ij} dx dy. \tag{10}$$

The corresponding numerical scheme is written down in terms of excess quantities (Berezovski et al., 2008). These quantities, however, are not constants but vary along the corresponding boundary. Therefore, the necessary step is to determine values of excess quantities.

4. Determination of excess quantities

Averaged values of excess quantities are determined exactly by means of jump relations at boundaries between computational cells, which express the continuity of true stresses and velocities (Berezovski et al., 2008). It should be noted that each corner of the computational cell $C_{n\,m} = [x_n, x_{n+1}] \times [y_m, y_{m+1}]$ can be considered as the central point of one of the corresponding four virtual cells $C_{n\pm1/2\,m\pm1/2} = [x_{n\pm1/2}, x_{n+1\pm1/2}] \times [y_{m\pm1/2}, y_{m+1\pm1/2}]$. In the first approximation, the value of every field quantity at corners of computational cells can be represented as the simple average of the corresponding values in neighbouring cells

To be able to perform the calculation of a particular problem we need to specify initial and boundary conditions. Initial conditions fix the state of each cell at a chosen time instant. Boundary conditions should be expressed in terms of averaged and excess quantities used in the numerical scheme. We expect that the state of cells adjacent to each boundary of the computational domain is known (at least partly). For the proper computing, we need to know in advance as many values of averaged and excess quantities as possible. All the boundary conditions follow from the jump relations at interfaces between cells and their surroundings (Berezovski et al., 2008).

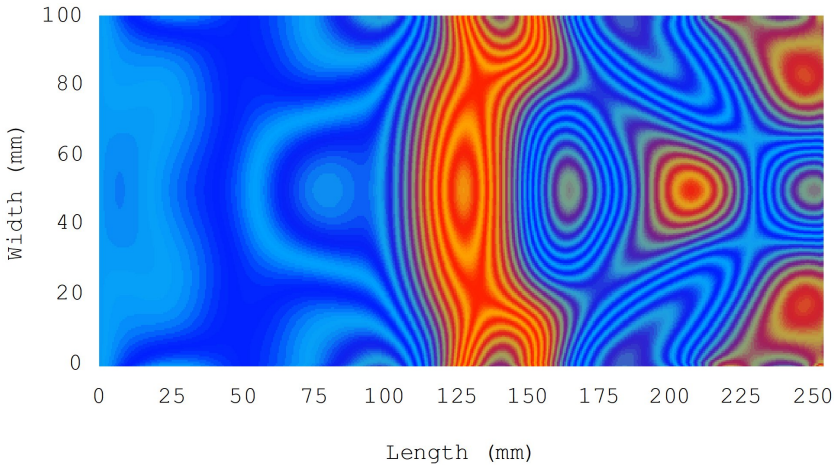

Fig. 1: Typical contour plot for pulse propagation after reflection at right boundary.

5. Test problem

As an example, a stress pulse propagation in a waveguide is considered. The length of the waveguide is 250 mm, its thickness is 100 mm. Calculations are performed for Al 6061 alloy characterized by the density 2700 kg/m^3, the Young modulus 68.9 GPa, and the Poisson ratio 0.33. This corresponds to the longitudinal wave velocity 5092 m/s. Choosing the space step equal to 1 mm, we have the time step 0.196 μs. The shape of the loading pulse at the left boundary is prescribed by $\sigma_{11}(t) = \sin^2(\pi t/80)$ for the first 80 time steps. After that the left boundary is stress-free. The right boundary is fixed and lateral boundaries are stress-free. The problem is solved by means of wave-propagation algorithm (LeVeque, 1997) and by means of the proposed numerical scheme. The main difference in these approaches is in the implementation of boundary conditions. In the wave-propagation algorithm (LeVeque, 2002), the boundary conditions are satisfied using the additional "ghost cells". In the proposed thermodynamically consistent scheme, the boundary conditions are imposed in terms of excess quantities at boundaries. It should be noted that in

the case of plane wave results of calculations obtained by both methods are identical. For non-plane wave, the distribution of longitudinal stress shows a similarity of results obtained by the two numerical methods. However, this similarity is not complete especially after reflection. The details of fields distribution depend on the implementation of boundary conditions in the pulse propagation in elastic waveguides.

6. Conclusions

The propagation of a pulse in elastic waveguides displays the result of interactions of distinct modes. Theoretically, only certain first modes are taken into account. Direct numerical simulation combines all of them by default. However, the implementation of boundary conditions should be as accurate as possible. In the paper, such an implementation is proposed in terms of excess quantities taken directly at boundaries.

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