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STOCHASTIC RESONANCE IN DYNAMICS AND RELATED DISCIPLINES

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Abstract: *Stochastic resonance (SR) is a phenomenon which can be observed in some nonlinear dynamic systems under combined excitation including deterministic harmonic force and random noise. This phenomenon was observed the first in the early 1940s when investigating the Brownian motion. Later several disciplines in optics, plasma physics, biomedicine and social sciences encountered effects of this type. However, the actual discovery and start of intensive period of investigation is dated in early 1980s when the idea of SR initiated remarkable inter disciplinary interest including most areas of physics, chemistry and neuro-physiology with a significant overlap to engineering and industrial area. Promising opportunities to employ SR in mechanics emerged only recently to model certain post-critical effects in non-linear dynamics and simultaneously to develop new vibration damping devices, energy harvesting facilities, sophisticated measuring technics and others. The aim of the paper is to present information about a new challenging discipline offering a large field of basic research and possibilities for practical applications.*

Keywords: Stochastic resonance, interwell hopping, non-linear vibration, Fokker-Planck equation.

1. Introduction

The Stochastic Resonance (SR) is a phenomenon which can be observed at certain non-linear dynamic systems under combined excitation including mostly deterministic periodic force and random noise. The phenomenon of this type has been first observed and reported by Kramers (1940), investigating the interwell hopping in the Brownian motion. Some allusions can also be found in older resources devoted to stochastic processes and theory of stability (Lyapunov, Kolmogorov, Planck and others).

The genuine phenomenon of SR has been discovered in early 1980s. The initiation point were probably two papers by C. Nicolis (1981, 1982) dealing with problems of climatic evolution. Other scientific and application areas followed that inspiration in due time, since it came to light that SR is a generic phenomenon. The idea of SR initiated remarkable cross disciplinary interest bringing together nonlinear dynamics, statistical physics, information and communication theories, data analysis, life and medical sciences. Individual areas came to the use of SR phenomenon rather independently and, therefore, they introduced slightly different definitions and particular strategies in the first period. This transition time passed and many cross disciplines overlapping in their activities have been build at the unifying background developed by mathematics and theoretical physics. Despite this evolution the historical aspects are still visible, due to fact that every branch still focuses on different needs, working in different scale and parameter intervals.

The notation Stochastic Resonance was introduced probably in 1981 in informatics to describe the annoying noise in communication equipment that prevented to detect a weak useful signal. However, researchers recognized soon that under certain conditions the noise can be helpful to enhance the device sensitivity.

The opportunity to employ SR in mechanics emerged only recently. SR approved to be promising for modeling of certain post-critical effects in non-linear dynamics, active vibration damping, feedback systems,

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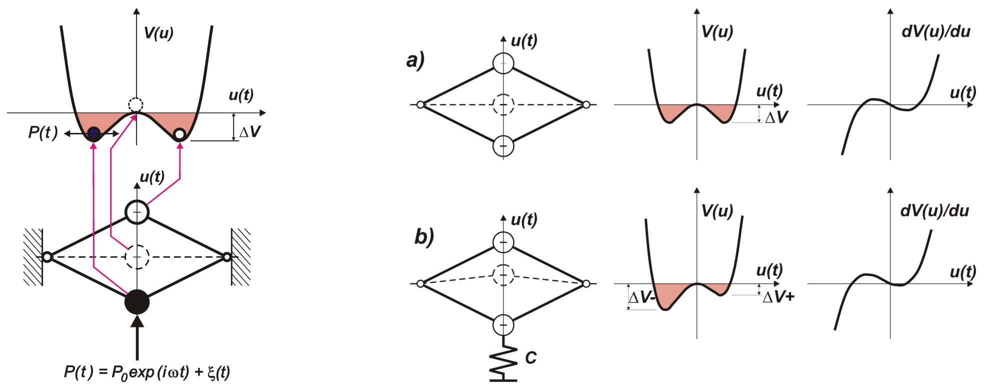

Fig. 1: Bi-stable nonlinear system: a) Symmetric potential; b) Non-symmetric potential.

biomechanics, etc. Therefore, it is worthy of presentation a certain overview to the community of rational and applied dynamics concerning strengths, weaknesses and application possibilities of SR occurred in theoretical and applied disciplines.

The phenomenon itself manifests in the simplest case by a stable periodic hopping between two nearly constant limits perturbed by random noises. The occurrence of this phenomenon depends on certain combinations of input parameters, which can be determined theoretically and verified experimentally. The conventional version of SR can occur in a bi-stable system under suitable combination of the additive Gaussian white noise and harmonic deterministic force. The classical mathematical definition of SR follows from properties of the Duffing equation with the negative linear part of the stiffness (bi-stable system) under excitation by a Gaussian white noise together with a deterministic harmonic force with a fixed frequency. It should be highlighted that also more general definitions of SR exist and will be also briefly reported in this paper. In particular, it considers various types of the random noise, shapes of the deterministic excitation component, types of oscillator non-linearity (potential of internal forces) and, finally, also a number of stable positions, which can exceed two or drop to one.

In terms of classical Engineering Dynamics, SR can be assumed as a dangerous effect accompanying a post-critical system response. Therefore, it should be eliminated by an appropriate selection of parameters and operating conditions (plasma physics, aeroelasticity, rotating machines, etc.) in order to ensure the reliability of the system. On the other hand, SR can characterize the mode of a natural system we are observing and, therefore, it serves as a tool of its investigation (e.g., Brownian motion mentioned above). It can also represent an intentional operating mode of the artificial system and in this case it should be considered as a useful state (special excitation or vibration damping devices, energy harvesting, etc.).

Nevertheless, many disciplines predominantly consider SR as a mechanism by which a system embedded in a noisy environment acquires an enhanced sensitivity towards small external signal, when the noise intensity reaches certain finite level. This phenomenon of boosting undetectable signals by resonating with added noise extends to many other systems, whether electromagnetic, physical or biological, and is an area of intense research. This interpretation of SR shows that noise can play a positive role in systems either designed artificially or observed as a natural systems. Furthermore, SR and its variants can serve to understand many processes in various scales and temperature domains to understand various effects in solid state physics, biophysics and electronics with possible application to design the SR inspired devices.

The study tries to mimic some excellent very well known review studies published mainly in the areas of physics, informatics and physiology with emphasis on Engineering Dynamics. See for instance papers: (Gammaitoni et al., 1998; Nicolis et al., 1993; Wellens et al., 2004; Luchinsky et al., 1999a,b; Anton and Sodano, 2007; Moss et al., 2004), etc. Although their style is quite different, adequately with the branch they represent, they are full of valuable information and worthy to be studied. For reading are recommended problem oriented monographs, e.g., (McDonnell et al., 2008; Tuckwell, 1988) or books including SR devoted chapter, e.g., (Moss, 1994; Berglund and Gentz, 2006; Anishchenko et al., 2003). Additional information can be found also at numerous web sites, like popular Wikipedia, Scholarpedia, American Physical Society Sites, Encyclopedia of Maths, or MathWorld, see (Weisstein, 2010). Doubtlessly the largest source of primary information are leading journals edited by world societies of physics, electronics, informatics and neuro sciences.

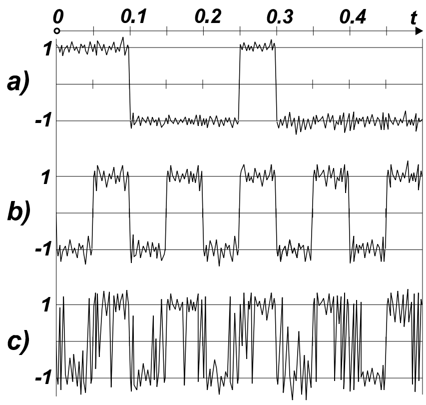

Fig. 2: Time history of the system response for various noise variance: (a) low level; (b) optimal level σ_0^2; (c) high level.

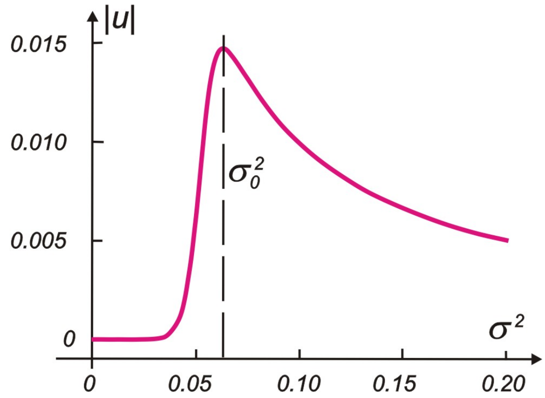

Fig. 3: Amplitude of the system response alternating component due to simultaneous excitation by a weak periodic force and a random noise.

2. Classical definition of Stochastic Resonance

In the classical meaning, SR occurs in bi-stable systems with Single Degree of Freedom (SDOF), when a small periodic force is applied together with a large broad band random noise, see Fig. 1. The system response is driven by two excitation components resulting in a "system switch" between two stable states. Their positions are given by two wells of the system potential $V(u)$. Wells are separated by a barrier. Its height, decisive for the switching, is considered as a difference between maximum and minimum of the potential. For the symmetric potential it can be noted ($\Delta V(u) = \Delta V_-(u) = \Delta V_+(u)$), see Fig. 1 right.

In the absence of periodic forcing, the approximate frequency of escape from one well into the second is given by the following estimate published in the comprehensive study due to Kramers (1940):

$$\omega_e = \sqrt{2} \cdot \exp(-\Delta V/\sigma^2) \tag{1}$$

where σ^2 is the variance of the noise and ΔV means the barrier separating potential minima (symmetric potential), see Fig. 1a. In classical setting of SR the Gaussian white noise is taken into account.

If both component are acting, then the degree of switching is related with the noise intensity σ^2, see a sample response in Fig. 2. When the periodic force is small enough being unable to make the system response switch, the presence of a non-negligible random component is required for it to happen. When the noise is small (small variance σ^2) very few switches occur, mainly at random with no significant periodicity in the system response - picture (a). When the noise is too strong a large number of switches occur for each period of the periodic component and the system response does not show remarkable periodicity - picture (c). Between these two conditions, there exists an optimal value of the noise intensity σ_0^2 that cooperatively concurs with the periodic forcing in order to make almost exactly one switch per period (a maximum in the signal-to-noise ratio) - picture (b). Amplitude of the response alternating component as a function of the noise level is outlined in Fig. 3. Peakness of the maximum is given by the damping factor. If the damping is too high, the peak can completely disappear and SR vanishes.

The optimum of the noise level σ_0^2 is quantitatively determined by the matching of two time scales:

(i) the period of the sinusoid (the deterministic time scale);
(ii) the Kramers rate, Eq. (1) - average switch rate induced by the sole noise which is the inverse of the stochastic time scale. It implicates the denomination "Stochastic Resonance".

The Kramers formula Eq. (1) is a result of theoretical and empirical investigation motivated by problems of nonlinear optics. Note that in original resources the absolute temperature T instead of the variance σ^2 is considered. The formula Eq. (1) is widely used and works very well. During the last decades, a number of areas of optics, quantum mechanics, chemistry, neurophysiology, etc. investigated this formula attempting to use the phenomenon of SR for description of various effects arising in their branches using both experimental and theoretical ways of investigation, see e.g., Inchiosa and Bulsara (1996); Dykman and et al. (1996).

The mathematical basis of the classical SR definition is related to the Duffing equation with negative linear part of the stiffness (in terms of mechanics). It is the most simple variant and it corresponds together with Gaussian white noise and deterministic harmonic force with a fixed frequency to the classical setting of SR. This configuration will be treated mostly throughout this paper.

Let us assume the nonlinear mass-unity SDOF oscillator written in a normal form:

$$\dot{u} = v; \quad \dot{v} = -2\omega_b \cdot v - V'(u) + P(t) + \xi(t). \tag{2}$$

$V(u)$ - potential commonly introduced in a form providing the Duffing equation:

$$V(u) = -\frac{\omega_0^2}{2}u^2 + \frac{\gamma^4}{4}u^4 \qquad \Rightarrow \qquad V'(u) = dV(u)/du = -\omega_0^2 \cdot u + \gamma^4 \cdot u^3 \tag{3}$$

$\xi(t)$ - Gaussian white noise of intensity $2\sigma^2$ respecting conditions:

$$\mathcal{E}\{\xi(t)\} = 0, \quad \mathcal{E}\{\xi(t)\xi(t')\} = 2\sigma^2 \cdot \delta(t - t'), \tag{4}$$

$\mathcal{E}\{\bullet\}, \delta(t)$ - operator of the mathematical mean value in Gaussian meaning or Dirac function, respectively,

$P(t) = P_o \exp(\mathrm{i}\Omega t)$ - external harmonic force with frequency Ω. Amplitude P_o should be understood per unit mass.

Symbols ω_0 and ω_b have a usual meaning of the circular eigen-frequency and circular damping frequency of the associated linear system. The linear part of the $V'(u)$ is negative making the system meta-stable in the origin, while the cubic part acts as stabilizing factor beyond a certain interval of displacement u. The system is drafted in the Fig. 1 in two versions: (a) system with symmetric potential typical by an equivalent energy needed for hopping from the left into the right potential well and backwards; (b) system with asymmetric potential due to the supplementary linear string which could be able (when rising its stiffness) to bring the oscillator to mono-stable state.

3. Methods of Stochastic Resonance investigation

Theoretical approaches either analytical or numerical are mostly based on an assumption that random processes ruling inside the system investigated are of the Markov type. The primary requirement, namely the dependence of the process on its value only in one previous moment is usually accomplished. In such a case, a large variety of methods are applicable for investigation of SR phenomena.

Basically three type of solution procedures can be regarded:

(i) Fokker-Planck (FP) Equation. It is the equation for cross Probability Density Function (PDF) of the system response. Solution of this equation serves subsequently for evaluation of various stochastic parameters like mean value, stochastic moments of adequate order, auto and cross correlation functions, probability flow, signal to noise ratio, mutual information etc. Concerning SR itself, the main indicators and parameters of this phenomenon can be evaluated and discussed in relation with physical character of the problem. So that PDF is a certain "source function" to obtain all information needed.

Taking into account that the random noise in the governing physical differential system, Eq. (2), has an additive character, no Wong-Zakai correction terms emerge, see e.g., (Wong and Zakai, 1965; Lin and Cai, 1995; Náprstek, 2003). Then the relevant FP equation, e.g., (Pugachev and Sinitsyn, 1987), can be easily written out:

$$\frac{\partial p(u,v,t)}{\partial t} = -\kappa_u \frac{\partial p(u,v,t)}{\partial u} + \frac{\partial}{\partial v}(\kappa_v p(u,v,t)) + \frac{1}{2}\kappa_{vv}\frac{\partial^2 p(u,v,t)}{\partial v^2}, \tag{5}$$

$$\begin{array}{lll} \kappa_u, \kappa_v \text{ - are drifft coefficients:} & \kappa_u = v \; ; & \kappa_v = \kappa_v(t) = -2\omega_b \cdot v - V'(u) + P(t) \; , \\ \kappa_{vv} \text{ - is a diffusion coefficient:} & \kappa_{vv} = 2\sigma^2, & \end{array} \tag{6}$$

together with boundary and initial conditions:

$$\lim_{u,v \to \pm\infty} p(u,v,t) = 0 \;\; (a), \qquad p(u,v,0) = \delta(u,v) \;\; (b). \tag{7}$$

Solution of the above FP equation can be conducted using one of the following procedures:

(i-a) Variational solution of Galerkin type. In principle it is a procedure of decomposition into stochastic moments (or cumulants) with Gaussian closure, e.g., (Kang et al., 2003).

In general, for details of the Galerkin method on the basis of functional analysis rules, see e.g., (Mikhlin, 1970). Details of particular solution see (Náprstek, 1996; Cai and Lin, 1988; Zhu et al., 1990), and other papers and monographs. The method is suitable namely for stationary solutions, but quasi-periodic solutions can be investigated as well, see, e.g., (Náprstek et al., 2015).

(i-b) Generalized Fourier method. Decomposition into a series eigen functions and values of FP operator.

$$p(u,v,t) = p_o(u,v) \cdot \varphi(t) \quad \Rightarrow \quad p(u,v,t) = \sum_{j=0}^{N} p_j(u,v) \cdot \varphi_j(t) \tag{8}$$

The series Eq. (8) can be substituted into the FPE Eq. (5). Due to the independence of $p_j(u,v)$ or $\varphi_j(t)$ on time or space variables, respectively, the part dependent on time only can be separated on the left side and that dependent on space variables on the right side. They can be equivalent only if both of them equal the same constant λ_j for each part of the series. It can be shown that λ_j are eigen values of the FP operator part which is on the right side of Eq. (5). Subsequently, $p_j(u,v)$ are relevant eigen functions of this operator and, finally, $\varphi_j(t)$ are the simple exponential functions with the negative real part. Take a note that the $\lambda_0 = 0$, as the first part of the series Eq. (8) for $j = 0$ represents the stationary part of the FPE solution, provided the stationary part exists. In general, the occurrence of one or more positive real parts of λ_j can reveal positive which would indicate an unstable solution of FPE. However, it is not the case when investigating FPE used for modeling the SR phenomenon.

This approach is applicable rather in special cases with easy searching of eigen functions, when transition process is looked for. For examples, see (Grasman and van Herwaarden, 1999). In general, to look for eigen functions of FP operator is complex and can prevent application of this method when more than SDOF system is analyzed.

(i-c) Floquet theory. Application of the Floquet theorem:

$$p(u,v,t) = p(u,v,t+T) \tag{9}$$

Suitable for equations with periodically variable coefficients, when transition non-periodic process is investigated. See (Grasman and van Herwaarden, 1999).

(i-d) Finite Element Method and other numerical procedures. The FEM can be considered as a general numerical solution method of partial differential equation. It can be proved that FEM is well applicable for this purpose under certain circumstances, which are fulfilled regarding FPE. When constructing adequate elements, a care should be taken due to special properties of the FP operator. Significant problem originates from the fact of multi-dimensionality of space we are working with and a delicate character of initial conditions. Moreover, the non self-adjointness of the FP operator, special configuration of boundary conditions, etc., should be taken into account. These factors shift application of FME in this case into a special area where a number of non-conventional problems should be solved.

The FPE is analyzed in an original evolutionary form which enables an analysis of transition effects starting the (nearly) Dirac type initial conditions. The FEM efficiency when solving FPE which follows from the Duffing stochastic differential equation without external harmonic forces was already studied by the authors in (Náprstek and Král, 2014). With the periodic force taken into account, certain difficulties arise due to the time inhomogeneity of the corresponding stochastic process. Many results regarding FEM application on FP equation analysis can be found in (Spencer and Bergman, 1993) or (Bergman et al., 1996). For the most recent results concerning FEM application to SR problem, see (Náprstek and Král, 2014), and additional details together with demonstrating examples can be found, e.g., in (Náprstek and Král, 2017).

The method is based on the approximaltion solution of Eq. (5) in the Galerin-Petrov meaning on the piecewise smoothly bounded domain $\Psi \in u \times v$, in $\mathbb{R}^d, d = 2$. The initial conditions at $t = 0\,\mathrm{s}$ for PDF are considered in a form of the Gauss distribution function with an initial system position at the point $u_0 = 0, v_0 = 0$. For a small values of standard deviation it approaches the Dirac function as it is primarily requested.

After a spatial discretizating of Ψ into the rectangular finite elements using the bilinear approximation functions and implying boundary condition $p(\partial\Psi, t) = 0$, the system of ordinary differential equations emerges with global matrices $\mathbf{M}$, $\mathbf{S}(t)$ and vector of probability density values $\mathbf{P}(t)$ in nodes of the mesh.

The final differential system has the form as follows:

$$\mathbf{M} \cdot \dot{\mathbf{P}}(t) \; = \; \mathbf{S}(t) \cdot \mathbf{P}(t) \tag{10}$$

The matrix $\mathbf{S}(t)$ is time-dependent due to the periodic perturbation entering the drift term of FPE and, in the result, the solution oscillates periodically between the potential wells. In the regime of SR, the switchings are in phase with the external periodic signal $\mathbf{P}(t)$ and the mean residence time is most close to half the signal period $2\pi/\Omega$. Comparison of results obtained by means of FME with those following from analytical investigation show quite well compatibility.

The efficiency of FEM is obvious as usual. It enables to investigate details which are inaccessible using other methods. It applies especially to transition processes starting the excitation and response processes nearby the stability loss, when the Lyapunov exponent is floating around zero and boundary between local and global stability are ambiguous.

(ii) Stochastic simulation - digital and analog. Stochastic simulation is one of the most important methods of SR investigation. The basic idea is straightforward, the governing system Eq. (2) is subdued to numerical integration and subsequently the probabilistic parameters including PDF are evaluated. However, the extreme caution should be taken as the differential system is stochastic. Because the system Eq. (2) includes only an additive noise, no Wong-Zakai correction terms are necessary, see (Wong and Zakai, 1965; Lin and Cai, 1995; Náprstek, 2003). However, the strategy of integration should be carefully controlled (Kloeden and Platen, 1992) due to fact that we manipulate with the Ito system. In principle the time increment can be neither too long in order to prevent information loss, nor too short to keep the stochastic character of the output. Hence the care should be taken during manipulations in the corrector phase of one step.

The results obtained in this manner are very important. They serve as a verification of semi-analytical results obtained using one of procedures mentioned in previous paragraph (i) and, furthermore, the simulation is able to enter into small details which remain hidden to methods mentioned in (i). It applies particularly to transition process if there is a need of their investigation. On the other hand, like every fully numerical method, the simulation technique provides the result for one set of parameters only. To obtain a larger overview is difficult and laborious (similarly like in experiments).

Analog simulations have been very popular in the past wherever nonlinear differential equations were to be solved. However, they are still very attractive for researchers as they lie at the frontier between digital simulation and experiment. Their advantage is that the parameters can be easily and quickly tuned over a wide range of values and the response can be followed straightforwardly. Many review and technical papers have been published as for instance (Gang et al., 1991; Gammaitoni et al., 1989), where a comparison of analog simulation of stochastic resonance with adiabatic theory has been performed. It should be appreciated now that a genuine analog simulation can be effectively emulated at digital computers using software packages, see, for instance, the McSimAPN package provided by McCann (2014). Actually, whatever hybrid analysis enabling digital support of the analog simulation is possible.

(iii) Experimental measurements. SR has been observed in a wide variety of experiments involving electronic circuits, chemical reactions, semiconductor devices, nonlinear optical systems, magnetic systems and superconducting quantum interference devices (SQUID). The general instructions are individually developed respecting specific character of the research activity. Anyway, be aware that the purpose of many experiments is an initial recognition of the basic principle while the theoretical approach should verify subsequently its validity. It is very frequently observed particularly in neurophysiological experiments related with SR, see the monograph by Tuckwell (1988) and papers (Ohka et al., 2012; Kosko and Mitaim, 2001; Mitaim and Kosko, 1998; Tanaka et al., 2003; McDonnell and Ward, 2011) and others. Three popular examples of this type performed should be named: the mechanoreceptor cells of crayfish, the sensory hair cells of cricket, human visual perception. Another "inverse" experiments (preceding any theoretical modeling) can be seen in a wind tunnel. Here the divergence instability of the prismatic bar in a cross flow has been observed in view of SR without any previous theoretical background. A number of primary experimental studies are available also in plasma physics, optics and in other branches, e.g., (Dinklage et al., 1999; McNamara et al., 1988; Gingl et al., 1995).

4. Conclusion

The article tried to indicate the essence of SR. This for the first view counterintuitive phenomenon brings a large impact on physical, biological and engineering systems. It is clear that SR is generic enough to be observable in a large variety of systems. The SR emerges in all scales, we can imagine. It governs the processes from nuclear fusion in the sun as far as the intra-atomic structures on the level of quantum mechanics. Amazing results of the basic research have been achieved and excellent industrial programs have been launched being based on many variants of SR. This concept of SR enabled to obtain an insight and exact description of many effects in macro and micro (nano) world and to fight successfully against various non-desirable phenomena in engineering. It resulted in many actually non-replaceable products of signal sensing and processing, medical instruments and treatment procedures. Many SR inspired neurophysiological implants represent cornerstones at the field.

The SR can be perceived as a natural phenomenon ruling inside certain dynamic systems. In such a case, it can act either positively as for instance to help stabilize the dynamic system and, therefore, to improve the system reliability or oppositely it can affect the system negatively, e.g., as a strong periodic exciting force, which is necessary to be avoided. The second view of SR understanding is considered in active synthesis and manipulation with the noise. Addition of the appropriate dose of (mostly) random noise onto the useful signal provides a significant increase of sensitivity and reliability of the equipment and enlarge its ability of data sensing, processing and possibly their usage in a feedback. The same is valid concerning an increase of information transfer capacity and reliability.

Let us be aware that SR is a challenging discipline for Engineering Dynamics offering a large variety of possibilities of new developments at theoretical as well as experimental platform. It could significantly enhance the top areas of nonlinear and stochastic dynamics closely related with Computational Mechanics, which is very advanced and widely used in comparison with other fields of numerical analysis. It provides strong support to Engineering Dynamics, which stands on the threshold to enter the field of research and application of SR.

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