23rd International Conference
ENGINEERING MECHANICS 2017
Svratka, Czech Republic, 15 – 18 May 2017

APPLICATION OF HAMILTONIAN MECHANICS IN EXPONENTIALLY STABLE CONTROL OF ROBOTS

V. Záda*

Abstract: *The paper deals with tracking control for robots-manipulators, where the dynamics is described by means of Hamiltonian mechanics. This way leads to different physical descriptive quantities used in control design. In the paper, the model-oriented Lyapunov-based control is considered. It is introduced in novel formulation by means of Hamiltonian mechanics.*

Keywords: Robotic manipulators, Hamiltonian mechanics, Robot dynamics, Robot control.

1. Introduction

Engineers usually use classical vector oriented Newton’s mechanics. Force interactions can be also described by scalar functions, recall Lagrangian or Hamiltonian formalism (Golstein, 1950), (Fasano, 2004). The majority of scientists use Lagrange’s equations for expression of robot dynamics (Siciliano, 2008), and robot control too. Generally, there are some limits for positions, velocities etc. These limits are constant for all configurations of robot (Arimoto, 1996). The state space, for control, is represented by positions and velocities, therefore by kinematic quantities. Momentums are not respected here. But, momentums change very quickly. Hence, the study of control methods may be interesting from Hamiltonian point of view too. Hamiltonian formalism with using a modified Hamiltonian was used as new function (Wen, 1988). A novel constructive method presented with a new Hamiltonian formulation in (Wang, 2005). The paper Teo (2013) presents a method for design of a set-point controller. The robot is described as a port-Hamiltonian system. In this contribution, we do not want continue similar way. We shall present a new exponentially stable method which may be used for electromechanical systems, especially for robot-manipulators.

2. Hamiltonian formalism

The basic ideas of Hamiltonian formalism and its using in the robot dynamics can be founded in the paper Záda (2016), therefore we shall omit them. The robot dynamics can be described by

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{1}$$

$$\dot{\mathbf{p}} = \mathbf{F} - \mathbf{g}(\mathbf{q}) - \left(\frac{\partial K(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}}\right)^T \tag{2}$$

The second form of these equations suitable for control of robot-manipulators can be represented by the set (Záda, 2016) of equations as follows

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{3}$$

$$\dot{\mathbf{p}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{p} - \mathbf{g}(\mathbf{q}) + \mathbf{u} \tag{4}$$

Now we return our attention to the problem of robot control.

* Assoc. Prof. Václav Záda, PhD.: Institute of Mechatronics and Engineering Informatics, Faculty of Mechatronics, Technical University of Liberec, Studentská 1402/2; 461 17, Liberec 1; CZ, vaclav.zada@tul.cz

3. Tracking control of robotic systems

The tracking control problem in the joint space consists of a given time-varying trajectory $\mathbf{q}_d(t)$ and its derivatives. The robot must follow this trajectory with sufficiently precision. Let us define vectors

$$\mathbf{e} = \mathbf{q} - \mathbf{q}_d \ , \ \ \mathbf{z} = \mathbf{M}\,(\dot{\mathbf{e}} - \mathbf{A}\mathbf{e})\ , \ \ \mathbf{y} = \mathbf{p} - \mathbf{z} \tag{5}$$

Let the controlled system be controlled by the control law, which hides model of controlled system

$$\mathbf{u} = \dot{\mathbf{y}} - \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{y} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{6}$$

So (5) and (6) represent our control system (model of controller). Matrices **A** and **B** are non-singular. They will be chosen lately. The vector **y** represents an estimation of the momentum **p,** and **z** is a difference between actual momentum **p** of the robot and **z**. If we use the equations (4) and (6) then we can derive a feedback equation for control process

$$\dot{\mathbf{z}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{z} - \mathbf{B}\,\mathbf{z} \tag{7}$$

Let a positive definite quadratic form be defined in the form

$$W = \frac{1}{2}\mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{z} \tag{8}$$

Then its time derivative along the trajectory given by (7) leads to the following inequality

$$\dot{W} = -\mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \ \le \ 0 \tag{9}$$

Generally, the multiplication of matrices in a quadratic form is positive definite. Hence the function W sinks in time. From (9) follows the chain of inequalities

$$0 \le \int_0^t \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z}\,dt = -\int_0^t \dot{W}\,dt = W(0) - W(t) \le W(0) \tag{10}$$

Let us define the smallest proper value of the matrix $\mathbf{M}^{-1}\mathbf{B}$

$$\lambda_{\min} = \min\left\{\lambda_q;\, \mathbf{M}^{-1}(\mathbf{q})\mathbf{B}\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}), \mathbf{q} \in Q\right\} \tag{11}$$

Where $\mathbf{x}(\mathbf{q})$, in (11), is a proper vector, Q is a working area of the controlled system. Then $\lambda_{\min} > 0$ and for all **z** it is valid

$$\lambda_{\min}\mathbf{z}^T\mathbf{z} \le \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \tag{12}$$

From these relations we derive the inequalities

$$\int_0^t \mathbf{z}^T\mathbf{z}\,dt = \frac{1}{\lambda_{\min}}\int_0^t \mathbf{z}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{z}\,dt \le \frac{W(0)}{\lambda_{\min}} < \infty \tag{13}$$

Since $\|\mathbf{z}\|^2 = \mathbf{z}^T\mathbf{z}$ and $0 \le W(t) \le W(0)$ we see that $\mathbf{z} \in L_2 \cap L_\infty$. Similarly let us define following values

$$\lambda_m = \min\left\{\lambda_q;\, \mathbf{M}(\mathbf{q})\,\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}), \mathbf{q} \in Q\right\} \tag{14}$$

$$\lambda_M = \max\left\{\lambda_q;\, \mathbf{M}(\mathbf{q})\,\mathbf{x}(\mathbf{q}) = \lambda_q\mathbf{x}(\mathbf{q}), \mathbf{q} \in Q\right\} \tag{15}$$

The numbers (14) and (15) are positive and hence the following inequalities (16) are valid too

$$\lambda_m\mathbf{z}^T\mathbf{z} \le \mathbf{z}^T\mathbf{M}\,\mathbf{z} \le \lambda_M\mathbf{z}^T\mathbf{z} \tag{16}$$

Remember, these inequalities of these quadratic forms are valid for all **z,** and, as usual, they can be simply written as

$$\lambda_m\mathbf{I} \le \mathbf{M}(\mathbf{q}) \le \lambda_M\mathbf{I} \tag{17}$$

The reader can prove that from (16) it can be derived the result

$$\lambda_M^{-1}\mathbf{I} \le \mathbf{M}^{-1}(\mathbf{q}) \le \lambda_m^{-1}\mathbf{I} \tag{18}$$

Then, we can obtain a new inequality

$$\frac{\dot{W}}{W} = -2\frac{\mathbf{z}^T \mathbf{M}^{-1}\mathbf{B}(\mathbf{q})\mathbf{z}}{\mathbf{z}^T \mathbf{M}^{-1}(\mathbf{q})\mathbf{z}} \le -2\frac{\mathbf{z}^T \mathbf{z}\,\lambda_{\min}}{\mathbf{z}^T \mathbf{z}\,\lambda_m^{-1}} \tag{19}$$

and so we have obtained the result

$$\frac{\dot{W}}{W} \le -2\lambda_{\min}\lambda_m \tag{20}$$

Le the multiplication of the proper values be denoted as $a = \lambda_{\min}\lambda_m$. The (20) may be rewritten as

$$\frac{\dot{W}}{W} \le -2a \tag{21}$$

Integration of (21) leads to the interesting estimation

$$W(t) \le W(0)e^{-2at} \tag{22}$$

Now, from (8) and (18) we can derive the following chain of inequalities

$$\lambda_M^{-1}\mathbf{z}^T\mathbf{z} \le \mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{z} = 2W \le 2W(0)\,e^{-2at} = \mathbf{z}^T(0)\mathbf{M}^{-1}(q_0)\,\mathbf{z}(0)\,e^{-2at} \le \|\mathbf{z}(0)\|^2\,\lambda_m^{-1}e^{-2at} \tag{23}$$

Hence, the variable **z** is exponentially bounded from above, how we see it from following inequality

$$\|\mathbf{z}(t)\| \le c_1 e^{-at}\text{, for } c_1 = (\lambda_M \lambda_m^{-1})^{0.5}\|\mathbf{z}(0)\| \tag{24}$$

The parameter c_1 depends only on initial estimation of the vector **z**(0). Now it can be seen that for $t\to\infty$ the variable $\mathbf{z}\to 0$. Let us study the differential equation (18) rewritten in the form

$$\dot{\mathbf{e}} - \mathbf{A}\mathbf{e} = \mathbf{M}^{-1}\mathbf{z} \tag{25}$$

This equation has a solution, for initial condition $\mathbf{e}_0 = \mathbf{e}(0)$,

$$\mathbf{e}(t) = \exp(\mathbf{A}t)\mathbf{e}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{M}^{-1}\mathbf{z}\,d\tau \tag{26}$$

Using classical inequalities for norms of matrices and vectors we can derive the following estimation

$$\|\mathbf{e}(t)\| \le c_2 e^{-bt} \tag{27}$$

where c_2 and b are some positive constants. Hence, for $t\to\infty$ the error vector $\mathbf{e}\to 0$, too.

Proof of (27): Let the solution (26) be considered. Then we obtain, with using norm for matrices,

$$\|\mathbf{e}(t)\| \le \|\exp(\mathbf{A}t)\|\cdot\|\mathbf{e}_0\| + \int_0^t \|\exp(\mathbf{A}(t-\tau))\|\cdot\left\|\mathbf{M}^{-1}\right\|\cdot\|\mathbf{z}\|\,d\tau \tag{28}$$

Because **A** is stable, there are positive constants k and c, such that for all $s \ge 0$ (here e is the Euler number) it is valid

$$\|\exp(\mathbf{A}s)\| \le ke^{-cs} \tag{29}$$

So we can write (e is the Euler number) the inequality

$$\|\mathbf{e}(t)\| \le k\cdot\|\mathbf{e}_0\|\cdot e^{-ct} + k\int_0^t e^{c(\tau-t)}\cdot\left\|\mathbf{M}^{-1}\right\|\cdot\|\mathbf{z}\|\,d\tau \tag{30}$$

The working space Q of all admissible vectors **q** is bounded, hence there are constant k_M such that

$$\left\|\mathbf{M}^{-1}(\mathbf{q})\right\| \le k_M \tag{31}$$

Hence with using these facts and (26) we can write

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}_0\|e^{-ct} + c_1 k k_M\, e^{-ct}\int_0^t e^{\tau(c-a)}d\tau \tag{32}$$

If c = a then the integral in (32) is t. Generally it is $c \ne a$. But the case c = a can be included in this general case, if we replace c by any smaller positive c. So we can rewrite (32)

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}_0\|e^{-ct} + c_1 k k_M \frac{e^{-at} - e^{-ct}}{c-a} \tag{33}$$

Let b = min{a, c}. Then

$$0 \le \frac{e^{-at} - e^{-ct}}{c-a} = \frac{e^{-bt}}{|c-a|} \tag{34}$$

and so (33) can be expressed in the following form

$$\|\mathbf{e}(t)\| \le e^{-bt}\left[k\|\mathbf{e}_0\| + \frac{c_1 k k_M}{|c-a|}\right] \tag{35}$$

Hence (35) has really the form of (27) for suitable constant c_2 . Let us rewrite the (25) in the form

$$\dot{\mathbf{e}} = \mathbf{M}^{-1}\mathbf{z} + \mathbf{A}\,\mathbf{e} \tag{36}$$

Then we obtain the following chain of inequalities

$$\|\dot{\mathbf{e}}\| \le \left\|\mathbf{M}^{-1}\right\| \cdot \|\mathbf{z}\| + \|\mathbf{A}\| \cdot \|\mathbf{e}\| \tag{37}$$

Now with using (24), (27) and (31), respectively, we obtain the following inequality with suitably constant c_3.

$$\|\dot{\mathbf{e}}(t)\| \le c_3 e^{-bt} \tag{38}$$

Here c_3 is the positive constant, too. Hence, if $t \to \infty$ then the signal, d**e**/dt, converges to 0. Relations (27) and (38) shove that the control algorithm is exponentially stable.

It is necessary to choice the matrix **A** to be stable. That is, its proper values must be in the left side of the complex plane. The matrix $\mathbf{M}^{-1}\mathbf{B}$ must be positive definite. Because **M** and so $\mathbf{M}^{-1}$ are positive definite matrices, it suffices to choice a matrix **B** to be diagonal with positive coefficients on diagonal. Then the multiplication $\mathbf{M}^{-1}\mathbf{B}$ is positive definite too. For simplicity, the matrix **B** can be chosen as $\mathbf{B} = b_0\mathbf{I}$, where b_0 is any function or constant, respectively. An alternative choice is to define a matrix $\mathbf{B}_0$ which is positive definite and then define matrix **B** as $\mathbf{B} = \mathbf{M}(\mathbf{q})\mathbf{B}_0$. Then the matrix $\mathbf{M}^{-1}\mathbf{B} = \mathbf{B}_0$ in is automatically positive definite.

4. Conclusions

In this article was developed the mathematically oriented text which describes the way how to control robots along a desired trajectory. The proved mathematical formulae show that the desired method is exponentially stable. This result is interesting for applications, because the exponential stability automatically leads to robustness of asked algorithm of control.

References

Arimoto, S. (1996) Control Theory of Non-linear Mechanical Systems. Clarendorf Press, Oxford.

Fasano, A. and Marmi, S. (2004) Analytical Mechanics. Oxford Press.

Golstein, H. (1950) Classical Mechanics, Addison-Wesley, Camb.

Samson, C. , Le Borgne, M. and Espiau, B. (1991) Robot Control, The task fun. approach. Clf. Press, Oxford.

Siciliano, B. and Khatib, O. (Eds.) (2008) Handbook of Robotícs. Springer.

Teo, Y.R., Donair, A. and Perez, T. (2013) Regulation and Integral Control of an Underactuated Robotic systems Using IDA-PBC with Dynamic Extension", IEEE/ASME International Conference of Advanced Intelligent mechatronics (AIM) Wollongong, Australia, pp. 920-925.

Wen, J.T. et al. (1988) New class of control laws for robotic manipulators. J. Control, Vol.47 (5), pp. 1361-1385.

Záda, V. and Belda, K. (2016) Robot control in terms of Hamiltonian mechanics. in: Proc. Int. Conf. on Engineering Mechanics 2016 (eds. Zolotarev, I and Radolf, V.), Svratka, Czech Republic, pp. 627-630.