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BEAM RESTED ON UNILATERAL ELASTIC FOUNDATION – (THEORY, EXPERIMENTS AND FINITE ELEMENT APPROACH)

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Abstract: *This work presents theory and numerical approach suitable for the solution of straight plane beams rested on an elastic unilateral (i.e. nonlinear modified Winkler's) foundation. The nonlinear reaction of the foundation can be described via nonlinear expression, in our particular case it is the positive part of the deflection function. The nonlinear differential equation of 4th-order is solved via standard Finite Element Method, which discretize the weak formulation of the problem. The Semi-smooth Newton's method is used to solve discrete problem.*

Keywords: Unilateral elastic foundation, Beam, Nonlinearity, Finite Element Method, Semi-smooth Newton's method.

1. Introduction

There are beams on elastic foundations which are frequently used in the technical practice; see Fig. 1a. In mechanics, the deflection $v = v(x)$ [m] of the beam without any volume loads is described by differential equation $EJ_{ZT}\frac{d^4v}{dx^4} + \mathrm{q_R} = 0$, where E [Pa] is the modulus of elasticity of the beam, J_{ZT} [m^4] is the major principal second moment of the beam cross-section and $\mathrm{q_R} = \mathrm{q_R}(x, v, \dots)$ [N.m^{-1}] corresponds to the reaction of the foundation (Frydrýšek et al., 2013; Frydrýšek et al., 2014); see Fig. 1b. Our work focuses on the solution of straight 2D beams of length 2L on an elastic foundation with nonlinear unilateral behaviour (linear Bernouli's beam, small deformations in the beam, Finite element Method); see Fig. 1. The methodology of the elastic foundation measuring applied in this paper is based on the pressing of a beam into the foundation (Klučka et al., 2014; Frydrýšek et al., 2014). on the displacement in the foundation can be approximated by nonlinear expression $\mathrm{q_R}(v) = \frac{k}{2}(|v| + v)$, see Fig. 1b.

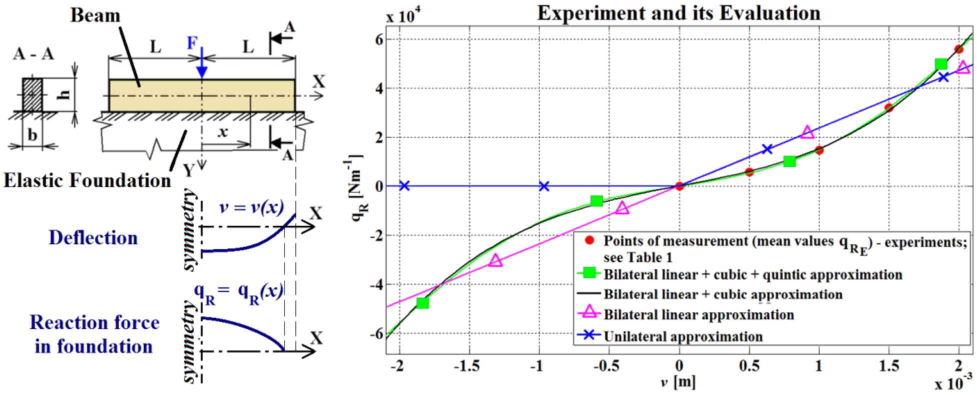

Fig. 1: a) Beam with cross-section b × h *and length* 2 L *is resting on elastic unilateral foundation; b) Dependence of reaction force on deflection (i.e. foundation load-settlement behavior for a sand, experiment and suitable linear and nonlinear approximations) (Frydrýšek et al., 2014).*

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For the sake of simplicity, the symbol v^+ is used instead of the expression $\frac{1}{2}(|v|+v)$ in this paper, i.e. $q_R(v) = k\frac{|v|+v}{2} = kv^+$, where the v^+ is known as the positive part of v.

2. Solved Example and its Boundary Conditions

Let us suppose that the solved beam has symmetry (i.e. the beam geometry, beam material, the volume and surface loads and the foundation are symmetrical). Therefore it is sufficient to solve the differential equation for a half of the beam, i.e. $x \in \langle 0;\ \mathrm{L}\rangle$, see Fig. 1a. Hence, the deflection of the beam is described by the equation $EJ_{ZT}\frac{d^4v}{dx^4} + kv^+ = 0$ for $x \in (0, \mathrm{L})$ with the following boundary conditions (prescribed in points x = 0 and x = L)

$$\frac{dv(x=0)}{dx} = 0, \quad (x = \mathrm{L}) = 0, \quad T(x=0) = -\frac{\mathrm{F}}{2}, \quad T(x=\mathrm{L}) = 0, \qquad (1)$$

where $T(x) = -EJ_{ZT}\frac{d^3v(x)}{dx^3}$ [N] is shearing force and $M_o(x) = -EJ_{ZT}\frac{d^2v(x)}{dx^2}$ [N.m] is bending moment.

3. Weak formulation of beam on unilateral foundation

Let's denote V as a space of virtual displacements and then $V = \left\{w \in H^2\big((0, \mathrm{L})\big): \frac{dw(x=0)}{dx} = 0\right\}$, where $H^2\big((0,\mathrm{L})\big)$ is Sobolev function space (i.e. Hilbert space with inner product $(v,w) = \int_0^{\mathrm{L}} \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + \frac{dv}{dx}\frac{dw}{dx} + v\,w\,dx$), see (Kufner et al., 1977). We multiply differential equation by virtual displacement $w \in V$ and integrate over the half length of the beam.

Then we obtain equation $EJ_{ZT}\int_0^{\mathrm{L}} \frac{d^4v}{dx^4} w\,dx + k\int_0^{\mathrm{L}} v^+ w\,dx = 0$, which is fulfilled for arbitrary $w \in V$. If we apply the integration by parts formula to the first integral and once more and subtract the boundary expressions from the left side to the right side we get

$$EJ_{ZT}\int_0^{\mathrm{L}} \frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\,dx + k\int_0^{\mathrm{L}} v^+ w\,dx = -\left[EJ_{ZT}\frac{d^3v}{dx^3}w\right]_0^{\mathrm{L}} + \left[EJ_{ZT}\frac{d^2v}{dx^2}\frac{dw}{dx}\right]_0^{\mathrm{L}}. \qquad (2)$$

From the boundary conditions (1) and from the property of $w \in V$ we obtain

$$-\left[EJ_{ZT}\frac{d^3v}{dx^3}w\right]_0^{\mathrm{L}} = -EJ_{ZT}\frac{d^3v(\mathrm{L})}{dx^3}w(\mathrm{L}) + EJ_{ZT}\frac{d^3v(0)}{dx^3}w(0) = \frac{\mathrm{F}}{2}w(0)$$

and $\left[EJ_{ZT}\frac{d^2v}{dx^2}\frac{dw}{dx}\right]_0^{\mathrm{L}} = EJ_{ZT}\frac{d^2v(L)}{dx^2}\frac{dw(L)}{dx} - EJ_{ZT}\frac{d^2v(0)}{dx^2}\frac{dw(0)}{dx} = 0$, and then the weak formulation of the beam defection on the unilateral foundation is following

find $v \in V$ such that $EJ_{ZT}\int_0^L \frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\,dx + k\int_0^L v^+ w\,dx = \frac{\mathrm{F}}{2}w(0)$ is fulfilled for all $w \in V$. (3)

The solvability (the existence of any solution) of (3) depends on the beam loads in general. In our particular case the necessary condition is inequality $\frac{\mathrm{F}}{2}p(x=0) > 0$, which is supposed to be fulfilled for arbitrary linear polynomial p positive on $(0,\mathrm{L})$. It means that the prescribed external force F must be positive. See (Sysala, 2008) for details.

4. Discretization by FEM

Let's divide the interval $(0,\mathrm{L})$ into n parts of the same length. This equidistant discretization with nodes $x_1 = 0, x_{i+1} = x_i + h$ has the constant step $h = \mathrm{L}/n$. The discrete approximation of the space V denoted by the symbol V_h is a subspace of the set of all smooth piecewise-cubic functions. Moreover, the first derivative of the every element of V_h is zero for $x = 0$. The choice of the space V_h stems from the convergence requirements of FEM theory, see (Haslinger, 1980) and from the mathematical embedding theory, see (Kufner et al., 1977). Follows that $V_h \subset V$. It is a finite dimensional space and therefore it has a basis, which is formed by piecewise-cubic functions

$$\varphi_{2i-1}(x)=\begin{cases}1-\frac{2(x-x_i)^3}{h^3}-\frac{3(x-x_i)^2}{h^2} & \text{for } x\in\langle x_{i-1},x_i\rangle,\\ 1+\frac{2(x-x_i)^3}{h^3}-\frac{3(x-x_i)^2}{h^2} & \text{for } x\in\langle x_i,x_{i+1}\rangle,\\ 0 & \text{otherwise,}\end{cases}\quad \varphi_{2i}(x)=\begin{cases}\frac{(x-x_i)^3}{h^2}+\frac{2(x-x_i)^2}{h}+(x-x_i) & \text{for } x\in\langle x_{i-1},x_i\rangle,\\ \frac{(x-x_i)^3}{h^2}-\frac{2(x-x_i)^2}{h}+(x-x_i) & \text{for } x\in\langle x_i,x_{i+1}\rangle,\\ 0 & \text{otherwise.}\end{cases}$$

It is obvious that the base functions are nonzero only on two subintervals of the beam discretization. The discrete form of (3) is following

find $v_h\in V_h$ such that $EJ_{ZT}\int_0^L \frac{d^2v_h}{dx^2}\frac{d^2\varphi_i}{dx^2}\,dx+k\int_0^L v_h^+\,\varphi_i\,dx=\frac{F}{2}\varphi_i(0)$ for all $i=1,\dots,2n+2$. (4)

Because the solution v_h of (4) is element of the space V_h, we can write $v_h=\sum_{i=1}^{2n+2}u_i\,\varphi_i(x)$. The values u_i for odd indexes are the deflections of v_h in the nodes of the discretization and values u_i for the even indexes are the slopes in nodes x_i and therefore $\boldsymbol{u}=\left(v_h(x_1),\frac{dv_h(x_1)}{dx},v_h(x_2),\frac{dv_h(x_2)}{dx},\dots v_h(x_{n+1}),\frac{dv_h(x_{n+1})}{dx}\right)^{\mathrm{T}}$. The algebraic FEM representation of the first integral in (4) and the right side of (4) can be set by a standart way and the local stiffness matrix on one subinterval of discretization can be derived

$$\boldsymbol{K_e}=\begin{pmatrix}12 & 6h & -12 & 6h\\ 6h & 4h^2 & -6h & 2h^2\\ -12 & -6h & 12 & -6h\\ 6h & 2h^2 & -6h & 4h^2\end{pmatrix}$$, see (Kolář et al., 1979).

The global stiffness matrix $\boldsymbol{K}$ and the global load vector $\boldsymbol{f}$ corresponding to (4) are shown (example for $n=4$, $h=\mathrm{L}/n$ constant).

$$\boldsymbol{K}=\frac{1}{h^3}\begin{pmatrix}12 & 0 & -12 & 6h & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -12 & 0 & 24 & 0 & -12 & 6h & 0 & 0 & 0 & 0\\ 6h & 0 & 0 & 8h^2 & -6h & 2h^2 & 0 & 0 & 0 & 0\\ 0 & 0 & -12 & -6h & 24 & 0 & -12 & 6h & 0 & 0\\ 0 & 0 & 6h & 2h^2 & 0 & 8h^2 & -6h & 2h^2 & 0 & 0\\ 0 & 0 & 0 & 0 & -12 & -6h & 24 & 0 & -12 & 6h\\ 0 & 0 & 0 & 0 & 6h & 2h^2 & 0 & 8h^2 & -6h & 2h^2\\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & -6h & 12 & -6h\\ 0 & 0 & 0 & 0 & 0 & 0 & 6h & 2h^2 & -6h & 4h^2\end{pmatrix},\quad \boldsymbol{f}=\begin{pmatrix}\frac{F}{2}\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\qquad(5)$$

5. The nonlinear mapping

We will present a way how to find an algebraic representation of the second integral in (4), which contains the nonlinear expression $v_h^+=(|v_h|+v_h)/2=(|\sum v_i\,\varphi_i|+\sum v_i\,\varphi_i)/2$. We first use the trapezoidal rule for approximating the integral $\int_0^{\mathrm{L}} v_h^+w_h\,dx$. The main goal of this is that we get approximation $|\sum_i v_i\varphi_i|\approx\sum_k|v_k|\varphi_k$, where the index i is form the set $\{1,2,3,\dots 2n+2\}$ and k is form $\{1,3,5,\dots 2n+1\}$. Now we get the homogenous equation $G(u)=0$ for $G(u)=EJ_{ZT}\boldsymbol{K}u+k\boldsymbol{B}u^+-\boldsymbol{f}$ instead of (4), where matrix $\boldsymbol{K}$ and vector $\boldsymbol{f}$ are from (5) and where the matrix $\boldsymbol{B}$ is diagonal, $\boldsymbol{B}=\mathrm{diag}(h/2,0,h,0,h,0,h,0,h/2,0)$.

Because we do not have available any derivation due to the absolute value in u^+, we cannot use the well-known Newton-Raphson's method. For this reason, we use semi-smooth Newton's method, see (Chen et al., 2001). This method introduces so called slanting function G^o and use it instead of Jacobian in the iterations. We define $G^o(u)=EJ_{ZT}\boldsymbol{K}+k\,\boldsymbol{B}\,\mathrm{diag}(A(u^+))$ in our case, where the symbol $A(u^+)$ stands for the active set of indexes of nods, in which subsoil is active. The resulting iterative equation in the (n+1)-th step is $u^{(n+1)}=u^{(n)}-G^o(u^{(n)})^{-1}\,G(u^{(n)})$ for known solution $u^{(n)}$ from the previous step. This iteration process converges for sufficiently small distance between the initial vector $u^{(0)}$ and the exact solution of the equation $G(u)=0$.

6. Results

The acquired FE results, see Fig. 2, were compared with the Central Difference Method too (Frydrýšek at al., 2014) with good agreement. There is a comparison of unilateral and bilateral approaches of elastic foundation in Fig. 2. For example:

$v_{MAX,bilateral} = 9.0607 \times 10^{-4}$ m, $M_{o\ MAX,bilateral} = 6.2070 \times 10^{4}$ N.m and $v_{MAX,unilateral} = 9.4242 \times 10^{-4}$ m, $M_{o\ MAX,unilateral} = 6.6701 \times 10^{4}$ N.m.

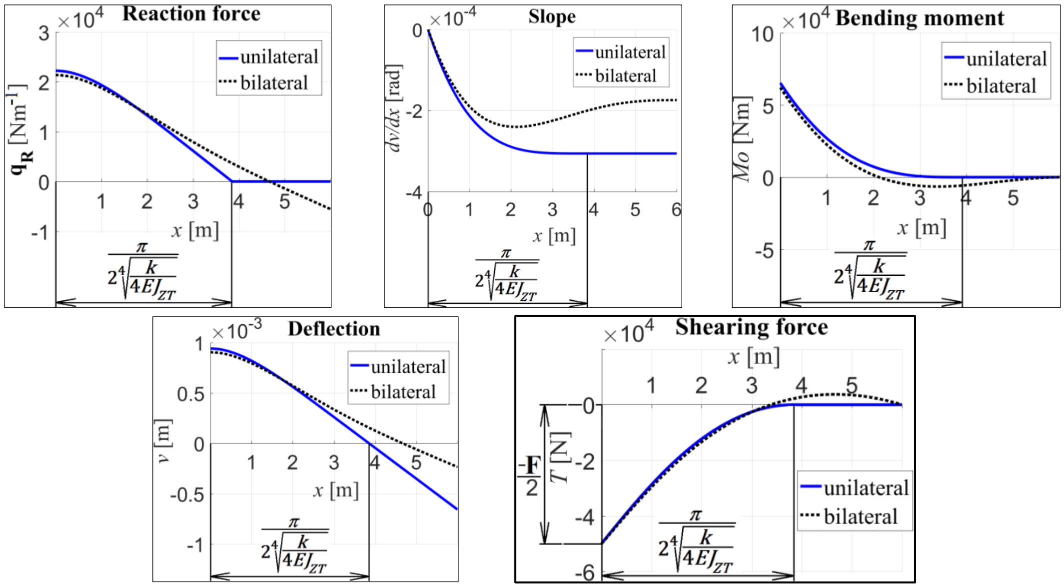

Fig. 2: Beam on unilateral and bilateral elastic foundation ($L = 6\ m,\ b = 0.2\ m,\ h = 0.4\ m,$ $F = 10^5\ N,\ E = 2 \times 10^{11}\ Pa,\ k = 2.3587 \times 10^7\ Pa$).

7. Conclusions

Straight beams on elastic foundations which nonlinear unilateral foundation were exposed and solved (i.e. theory, weak formulation, finite elements, semi-smooth Newton's method and results). The differences between unilateral and simple bilateral foundation are shown. Note, the beams on unilateral foundation cannot be solved via Newton's method but via semi-smooth Newton's method as shown in out article.

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