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THE APPLICATION OF INTEGRAL TRANSFORMS TO SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF THE FRACTIONAL ORDER

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Abstract: *The authors presented a description of the mathematical model of the string vibrations with the use of a partial differential equation of the fractional order. They applied the Caputo fractional derivative in their deliberations. The following integral transforms were used in order to solve this problem: the Laplace transform for t time and finite sine transform for x displacement.*

Keywords: Integral Transform, Fractional Wave Equation, Caputo Fractional Derivative, Mittag-Leffler Function.

1. Introduction

Over the last decades, partial differential equations of the fractional order have been applied in the modelling of various processes, such as energy propagation, thermal stresses, relaxation vibrations, nonlinear control system theory or robotics. Derivatives and integrals of fractional order became an attractive area of mathematics, being widely applied in various fields of science, especially physics, astrophysics, nuclear physics, chemistry, classical mechanics, quantum mechanics, vibration (Nowakowski, Miko, et al., 2016) or engineering (Blasiak, M., 2016; Laski, 2016; Laski et al., 2014; Nowakowski, Miesikowska, et al., 2016; Takosoglu et al., 2016; Zwierzchowski, 2016). A very important model is the fractional wave equations, which has been used in the theory of vibrations of smart materials in media where the memory effects cannot be neglected (Herzallah et al., 2010). The authors of papers (Blasiak, S., 2016; Stanislavsky, 2005) replaced the classical wave equation with the derivative of the fractional order.

This paper presents the fractional generalization of the wave equation describing the propagation of a wave in the string for the set initial and boundary conditions. The main advantage of the fractional calculation is the possibility of the investigation of the nonlocal response of mechanical systems, contrary to the classical calculation. The Caputo fractional derivative, as well as the Laplace transform for *t* time (Schiff, 1999) and finite sine transform for *x* displacement were applied in order to describe the problem in question.

2. Methods

The considerations apply to a uniform, flexible string of finite *l* length, with the height difference of mounting of two ends along *y* axis amounting to *a*. The physical properties of the system are determined by the following parameters: ρ - mass of the string section with a unit length and *T* - tensile strength. A wave causing vibrations perpendicular to the string length was taken into consideration. A case of low string vibrations, where the string deflections from the balance point are low in comparison with the string length, was considered.

The time-fractional wave equation (1) was presented below.

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$$\rho \frac{\partial^\alpha y}{\partial \tau^\alpha} = T \frac{\partial^2 y}{\partial x^2} \qquad 0 < x < l,\ t > 0 \tag{1}$$

By substituting $t = \frac{T}{\rho}\tau = v\tau$, the following equation was formulated:

$$\frac{\partial^\alpha y}{\partial t^\alpha} = \frac{\partial^2 y}{\partial x^2} \qquad 0 < x < l,\ t > 0 \tag{2}$$

The following boundary and initial conditions were considered:

$$\begin{aligned} &a)\ y(0,t) = 0,\ t > 0,\\ &b)\ y(l,t) = a,\ t > 0,\\ &c)\ y(x,0^+) = 0,\ 0 < x < l, 0 < \alpha \le 2,\\ &d)\ y_t(x,0^+) = 0,\ 0 < x < l, 1 < \alpha \le 2. \end{aligned} \tag{3}$$

By substituting the finite sine transform in the general form $f^*(n,t) = \int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$ to equation (2), the following equation was formulated:

$$\Im_s\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{n\pi}{l}\left[y(0,t) - (-1)^n y(l,t)\right] - \frac{n^2\pi^2}{l^2}y^* = \frac{n\pi}{l}\left[-(-1)^n a\right] - \frac{n^2\pi^2}{l^2}y^* \tag{4}$$

The left side in equation (2) describes the Caputo fractional derivative in the general form (Parsian, 2012):

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) \equiv D_C^\alpha f(t) \tag{5}$$

where:

$$D_C^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau \qquad t > 0, \qquad n-1 < \alpha < n \tag{6}$$

Γ - Euler gamma function.

the Laplace transform rule for fractional derivative

$$\mathscr{L}\left\{D_C^\alpha f(t)\right\} = s^\alpha \bar{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n \tag{7}$$

$$\mathscr{L}\left\{D_C^\alpha y(t)\right\} = s^\alpha \bar{y}(s) - \underbrace{y^{(0)}(0^+)s^{\alpha-1-0}}_{=0} - \underbrace{y^{(1)}(0^+)s^{\alpha-1-1}}_{=0}, \qquad n-1 < \alpha < n$$

Equation (2), after applying the Laplace transform and boundary condition (3)b $\bar{y}(l,s) = \frac{a}{s}$, might be recorded as follows:

$$\bar{y}^*(s) = -(-1)^n \frac{al}{n\pi}\left(\frac{1}{s} - \frac{s^{\alpha-1}}{(s^\alpha + \omega^2)}\right) \tag{8}$$

For the second element from the brackets in equation (8), the inverse Laplace transform was used:

$$\mathscr{L}^{-1}\left[\frac{s^{\alpha-1}}{\left(s^{\alpha}+\omega^{2}\right)}\right]=t^{1-1}E_{\alpha,1}\left(-\omega^{2}t^{\alpha}\right)=E_{\alpha}\left(-\omega^{2}t^{\alpha}\right)\ \text{for}\ \omega^{2}=\left(\frac{n\pi}{l}\right)^{2}$$

where $E_{\alpha,\beta}(z)$ means Mittag-Leffler type function defined by the series

$$E_{\alpha,\beta}(z)=\sum_{v=1}^{\infty}\frac{z^{v}}{\Gamma(\alpha v+\beta)}\qquad \alpha>0,\ \ \beta>0 \tag{9}$$

That is why, after applying the inverse Laplace transform, equation (8) takes the following form:

$$y^{*}(n)=-(-1)^{n}\frac{n\pi a}{l}\frac{1}{\omega^{2}}\left(1-E_{\alpha}\left(-\omega^{2}t^{\alpha}\right)\right) \tag{10}$$

After using the inverse sine transform, the following equation was obtained:

$$y(x,t)=-\frac{2a}{l}\sum_{n=1}^{\infty}\ (-1)^{n}\frac{l}{n\pi}\left(1-E_{\alpha}\left(-\omega^{2}t^{\alpha}\right)\right)\cdot\sin\left(\frac{n\pi x}{l}\right) \tag{11}$$

By using the following dependency:

$$\frac{2a}{l}\sum_{n=1}^{\infty}\frac{l}{n\pi}(-1)^{n}\cdot\sin\left(\frac{n\pi x}{l}\right)=-\frac{a\,x}{l}$$

The dependency describing the string vibrations takes its final form

$$y(x,\tau)=\frac{a\,x}{l}+\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}E_{\alpha}\left(-v\omega^{2}\tau^{\alpha}\right)\cdot\sin\left(\frac{n\pi x}{l}\right) \tag{12}$$

For $E_{2}(-z)=\cos\left(\sqrt{z}\right)$

$$y(x,\tau)=\frac{a\,x}{l}+\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}\cos\left(\sqrt{v}\,\frac{n\pi\tau}{l}\right)\cdot\sin\left(\frac{n\pi x}{l}\right) \tag{13}$$

Results

The following data was adopted for the calculations: the string length l = *20 m*, the height difference of mounting of two string ends a = *0.1 m*, the velocity of transverse waves v = *0.25 m/s*.

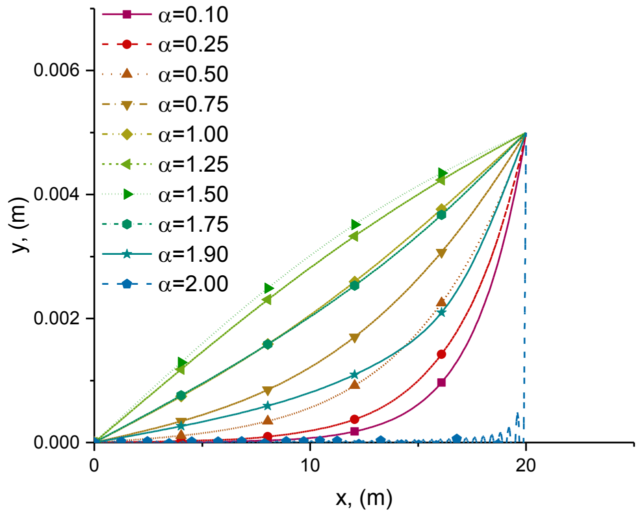

Fig. 1: Wave propagation in the string for different α .

3. Conclusions

This paper describes the mathematical model of string vibrations of a finite length with the use of a partial differential equation of the fractional order. The application of fractional calculus for the calculation of the presented model makes it possible to determine the whole set of the waveforms of the string vibrations, and not only as it is found in the classical theory of vibrations, equations 1 or 2. The Laplace transform for t time and finite sine transform for *x* displacement were applied in order to solve the presented model. The main advantage of using the integral transforms is the solution of partial differential equations by transforming them into algebraic equations as well as automatic consideration of the initial and boundary conditions in the solving process.

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