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COMPARISON OF DIFFERENT SIMULATION TECHNIQUES FOR RELIABILITY-BASED DESIGN OPTIMIZATION

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Abstract: *This contribution focuses on a double-looped reliability-based design optimization, in which the reliability of the system is evaluated in the inner loop and the designing process is performed in the outer loop. The double-looped formulation provides the most accurate results but it is computationally demanding especially if advanced simulation techniques are used for rare failure events. The selection of the method for the reliability assessment is therefore crucial to obtain the best results with the lowest possible computational efforts. A quasi-Monte Carlo simulation, an Asymptotic sampling and a Subset simulation are therefore utilized in the inner loop and the results are compared for two reliability-based design optimization benchmarks.*

Keywords: Multi-objective Optimization, Reliability-based Design Optimization, Subset Simulation, Monte Carlo Method, Asymptotic Sampling.

1. Introduction

A structural optimization is a process that seeks the best design under some predefined constraints. A deterministic model is usually unrealistic due to the uncertain inputs such as material properties, a structural topology, loadings etc. The optimal design with deterministic variables often terminates at a boundary between the failure domain and the safe domain and even a small perturbation in inputs can lead to a fatal failure. For that reason, the model uncertainties are introduced; the parameter uncertainties are associated with the input data whereas the structural uncertainties express that the model need not clearly describe the physics of the problem. The optimization under uncertainties looks into two main tasks; the first task (a *robust design optimization*) deals with the everyday fluctuations in inputs and provides a design with a minimum price that is less sensitive to small perturbations in inputs; the second task (a *reliability-based design optimization*) concentrates on worse-case scenarios and offers an economical design with large safety.

A reliability-based design optimization (RBDO) can be formulated by two linked loops. An optimizer provides a design in the outer loop, for which a probability of failure is evaluated in the inner loop. The double-looped procedure allows a very accurate safety appraisal of each design without any kind of approximation. However, this formulation suffers from large computational demands if a classical Monte Carlo method is used. Fortunately, advanced simulation techniques such as an Asymptotic sampling (Bucher, 2009) and a Subset simulation (Au & Beck, 2001) can be used for the reliability assessment and the accuracy can be almost maintained with the drastic computational effort reduction.

2. Multi-objective double-looped reliability-based design optimization

A classical formulation of the RBDO minimizes a cost function $f(\boldsymbol{d})$ such that the reliability constraints $\beta_i(\boldsymbol{x},\boldsymbol{d}) \geq \beta_i^{\min}$ as well as deterministic constraints $H_j(\boldsymbol{d}) \leq 0$ have to be satisfied. Design variables are arranged in vector $\boldsymbol{d}$ (e.g. deterministic variables or means of random variables), whereas uncertain parameters are arranged in vector $\boldsymbol{x}$. A generalized reliability index β_i for failure mode i is obtained by the inverse cumulative distribution function of the standard normal distribution $\beta = \Phi^{-1}(1 - p_f)$ where p_f is

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a probability of failure; $\beta_i^{\min}$ is the corresponding lower bound of the reliability index. In case that the setting of the lower limit $\beta_i^{\min}$ is unclear, the single-objective optimization can be reformulated into a multi-objective optimization; minimization of a cost function remains and maximization of the reliability indexes for all failure modes are added as the next objectives:

$$\min f(\boldsymbol{d}), \tag{1}$$

$$\max \beta_i(\boldsymbol{x}, \boldsymbol{d}),\, i = 1, \dots, n_I, \tag{2}$$

$$s.t. \quad H_j(\boldsymbol{d}) \leq 0, j = 1, \dots, n_J, \tag{3}$$

$$\boldsymbol{d}^{\min} \leq \boldsymbol{d} \leq \boldsymbol{d}^{\max}. \tag{4}$$

A probability of failure p_f is evaluated in an n-dimensional space of random variables $X_1, \dots, X_n$ as

$$p_f = \text{Prob}[F] = \text{Prob}[G(\boldsymbol{X}) \leq 0] = \int_{G(\boldsymbol{X}) \leq 0} f_X(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \tag{5}$$

where $f_X(\boldsymbol{x})$ is a joint probability density function, F denotes the failure, and $G(\boldsymbol{X})$ is a limit state function. A limit state $G(\boldsymbol{X}) = 0$ divides the space into the failure region $G(\boldsymbol{X}) \leq 0$ and the safe domain $G(\boldsymbol{X}) > 0$. A probability of failure is solvable analytically only for some specific problems and traditional numerical integration is not applicable for large n. Statistical sampling techniques based on a *Monte Carlo method* allow to estimate the expected value of p_f. Equation (5) is possible to reformulate as

$$p_f = \int_{-\infty}^{\infty} I_G(\boldsymbol{x}) f_X(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \tag{6}$$

where $I_G(\boldsymbol{x})$ is an indicator function that is equal to one for a failure domain and zero otherwise. For large number of samples, p_f in Equation (6) can be estimated by a ratio of the number of defective samples and the number of all samples. The number of total samples is recommended to set from $10/p_f$ with a coefficient of variation CV_{MC} around 30 %, over $100/p_f$ with CV_{MC} around 10 %, to $500/p_f$ with CV_{MC} around 5 %. In case that a sophisticated sampling strategy such as Halton or Sobol sequences or Latin hypercube sampling is used, the results are even more credible. The number of samples is however still enormous for small probabilities of failure.

A *Subset simulation* (Au & Beck, 2001) is based on a formulation of the failure event F as an intersection of the intermediate failure events F_i. The rare event problem is then reformulated as a series of more frequent events that are easier to solve. The probability of failure is as follows

$$p_f = \text{Prob}[F_1] \cdot \prod_{k=2}^{L} \text{Prob}[F_k \,|F_{k-1}]. \tag{7}$$

The failure probability of the first intermediate domain is evaluated by a classical Monte Carlo method with hundreds of samples N. These samples are sorted in an ascending order and a limit state function is shifted such as $\text{Prob}[F_1]$ is equal to a predefined value p_0. The first $(p_0 \cdot N)$ samples are used as seeds for the simulation of samples from conditional probabilities by a Markov chain Monte Carlo (MCMC) with modified Metropolis algorithm. In each level k, samples obtained by MCMC are sorted and first $(p_0 \cdot N)$ samples serve as seeds in k+1 step together with a proper shift of the limit state function. The last level L is reached if the probability of failure with the original limit state is greater than p_0.

An *Asymptotic sampling* (Bucher, 2009) is a novel methodology that predicts a reliability index from an asymptotic behavior of the probability of failure in an n-dimensional independent and identically distributed normal space. A principal idea is to sequentially scale random variables in a standard normal space over the standard deviation σ by a factor φ that is lesser than 1 to get more samples from a failure domain. In step k, scaled standard deviations σ_k equal to σ_{k-1}/φ are used to perform a Monte Carlo simulation with hundreds of samples. A corresponding reliability index β_k in step k and a factor φ raised to the power of k are saved as a support point for the following regression. After sufficient steps k, the approximation of the reliability index is obtained as a summation of the regression coefficients A and B after a regression via support points $\boldsymbol{\beta}$ and $\boldsymbol{\varphi}$

$$\boldsymbol{\beta} = A\boldsymbol{\varphi} + B\boldsymbol{\varphi}^{-1}. \tag{8}$$

A *First order reliability method* (FORM) (Hasofer & Lind, 1974) is not a simulation technique but an often used analytical approximation method in RBDO for its low computational demands. Nevertheless, it is inaccurate for highly nonlinear limit state functions. It is based on the linearization of the limit state function in a design point $\boldsymbol{u}^*$ in the standard normal space (SNS). The design point can be found by any optimization method as $\boldsymbol{u}^* = \min(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u})^{-1}$ subject to $\mathrm{G}(\mathrm{T}_{\mathrm{SNS}\to\mathrm{OS}}(\boldsymbol{u})) = 0$. A transformation from the standard normal space to the original space and vice versa is possible via Rossenblat transformation $\boldsymbol{x} = T_{SNS \rightarrow OS}(\boldsymbol{u})$. The approximation of the reliability index is then the shortest distance from the origin of SNS to the design point lying on the limit state surface.

3. Numerical benchmarks

Both benchmarks were optimized by Non-dominated sorting genetic algorithm II (Deb et al, 2002) with 200 individuals and 50 generations to obtain a rich approximation of the Pareto-set and the Pareto-front. The Pareto-front was bounded such that a reliability index is from an interval [0, 5.5] for both benchmarks. Both problems converged sufficiently in approximately the 10^{th} generation. An Asymptotic sampling as well as a Subset simulation was set to have CV_β equal to 5 % in the single objective optimum taken from literature with 1,000 independent runs.

The first example is taken from (Chen et al, 2013) and it is reformulated into a multi-objective optimization task in (Pospíšilová & Lepš 2015). The cost function is quadratic, the limit state function is highly nonlinear. The limit state $G(\boldsymbol{x}) = 0$ is also depicted in Fig. 1 with the solid line; the feasible domain is inside the shape. Both variables have the normal distribution and they are statistically independent. Means are design variables. Since a Monte Carlo method has high computational demands, an Asymptotic sampling was used to predict a probability of failure and the necessary number of samples was computed subsequently with $10/p_f$. The lower tail of the Pareto-front is almost identical for all used reliability assessment methods, FORM fails for reliability index slightly greater than 2.3 and it is problematical and unreliable for the rest of the Pareto-front. The Pareto-front with a Monte Carlo method oscillates between the front with an Asymptotic sampling and a Subset simulation approximately from reliability index equal to 4. At the upper tail, a Monte Carlo method is not reliable since these samples are in the failure domain according to the left figure in Fig. 1. The total number of samples during whole optimization process was measured in 10 independent runs: an Asymptotic sampling used ca. $2.3 \cdot 10^8$ samples, a Subset simulation used ca. $1.4 \cdot 10^8$ of samples, FORM in contrast used only $2.4 \cdot 10^5$ samples. A Monte Carlo method was used only once with enormous $5.4 \cdot 10^{11}$ samples leading to high CV.

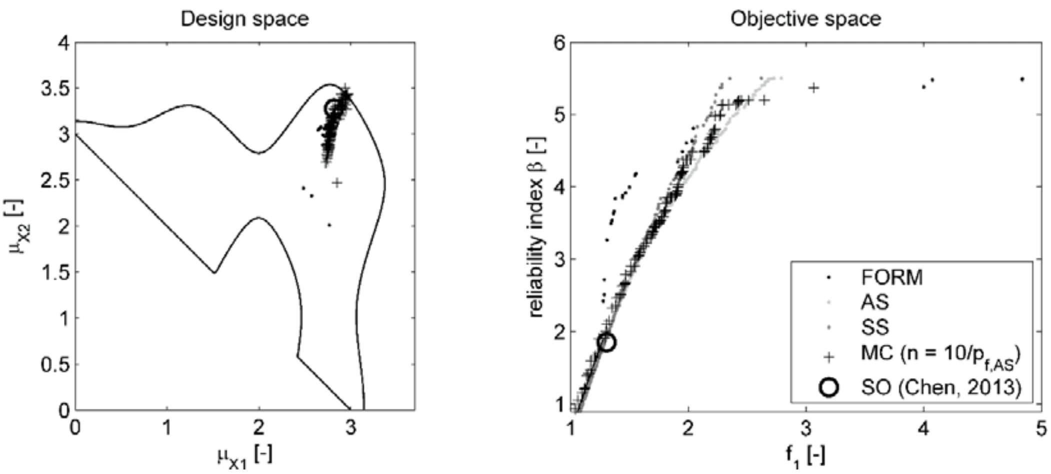
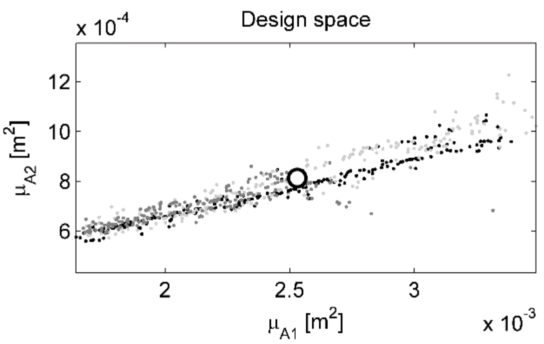


Fig. 1: Approximations of Pareto-set (left) and Pareto-front (right) for Example 1. Abbreviations: FORM – First order reliability method, AS – Asymptotic sampling, SS – Subset simulation, MC – Monte Carlo method, SO – single optimum from (Chen et al, 2013).

The second example is concentrated on a minimization of the material volume of a 23-bar planar truss bridge and maximization its safety. The limit state function is represented by a design rule that the mid-span deflection should not exceed 10 cm. The single objective formulation in (Dubourg, 2011) is reformulated into the multi-objective in (Pospíšilová & Lepš, 2013). Young's moduli E_1 and E_2 have Lognormal distribution; cross-sectional areas A_1 and A_2 have Lognormal distribution as well with means as design variables μ_{A1} and μ_{A2}; gravity loads P_1 - P_6 have a Gumbel distribution. All variables are statistically independent. A Subset simulation behaves the same as an Asymptotic sampling on interval [0, 3.1]. The rest of the Pareto-front with a Subset simulation probably needs more samples and/or levels and therefore an adaptive setting. FORM has a similar trend as an Asymptotic sampling but FORM slightly overestimates the reliability; this trend is obvious from the single optimum (Dubourg, 2011) lying on the Asymptotic sampling Pareto-front.



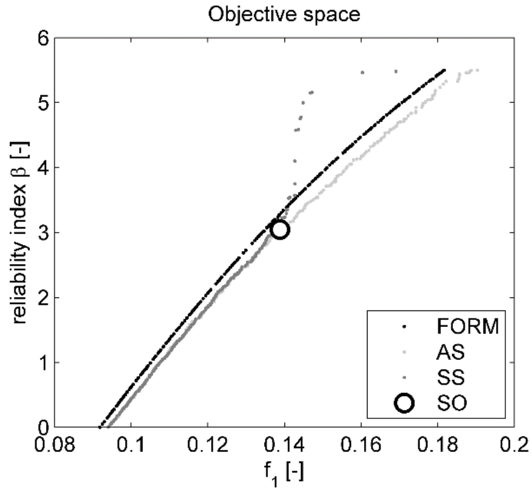


Fig. 2: Approximations of Pareto-set (left) and Pareto-front (right) for Example 2. Abbreviations: FORM – First order reliability method, AS – Asymptotic sampling, SS – Subset simulation, SO – single optimum from literature (Dubourg, 2011).

4. Conclusions

The multi-objective formulation of a reliability-based design optimization provides much more information than a single-objective case for a decision maker, who can subsequently decide which reliability level is worth it. On the other hand, computational efforts are much higher than for a single-objective formulation and therefore the most computationally demanding part, the reliability assessment, has to be chosen carefully. The double-looped formulation is affordable with advanced simulation techniques such as an Asymptotic sampling or a Subset simulation. Even with a basic implementation, the results are relatively credible with fewer number of samples than with a classical Monte Carlo method and more credible than with FORM.

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References

Au, S. K., & Beck, J. L. (2001). Estimation of small failure probabilities in high dimensions by subset simulation. Probabilistic Engineering Mechanics, 16, 4, pp. 263-277.

Bucher, C. (2009). Asymptotic sampling for high-dimensional reliability analysis. Probabilistic Engineering Mechanics, 24, 4, pp. 504-510.

Chen, Z., Qiu, H., Gao, L., & Li, P. (2013). An optimal shifting vector approach for efficient probabilistic design. Structural and Multidisciplinary Optimization, 47, 6, pp. 905-920.

Deb, K., Pratap, A., Agarwal, S., & Meyarivan, T. A. M. T. (2002). A fast and elitist multiobjective genetic algorithm: NSGA-II. Evolutionary Computation, IEEE Transactions on, 6, 2, pp. 182-197.

Dubourg, V. (2011). Adaptive surrogate models for reliability analysis and reliability-based design optimization PhD Thesis, Université Blaise Pascal-Clermont-Ferrand II.

Hasofer, A. M., & Lind, N. C. (1974). Exact and invariant second-moment code format. Journal of the Engineering Mechanics division, 100, 1, pp. 111-121.

Lee, T. H., & Jung, J. J. (2008). A sampling technique enhancing accuracy and efficiency of metamodel-based RBDO: Constraint boundary sampling. Computers & Structures, 86, 13, 1, pp. 463-1476.

Pospíšilová, A. & Lepš, M. (2013). Multi-Objective Optimization with Asymptotic Sampling for RBDO. In: Proceedings of the Third International Conference on Soft Computing Technology in Civil, Structural and Environmental Engineering.

Pospíšilová, A. & Lepš, M. (2015). Multi-Objective Reliability-Based Design Optimization utilizing an Adaptively Updated Surrogate Model. In: Proceedings of the Fourth International Conference on Soft Computing Technology in Civil, Structural and Environmental Engineering.