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ROOT-FINDING METHODS FOR SOLVING DISPERSION EQUATIONS

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Abstract: *Contribution deals with the methods of rootfinding in a plate and cylindrical waveguide. The methods are based on: classic root-finding, interval arithmetics, Chebyshev interpolation, marching squares and marching triangles. All methods have been implemented both in Matlab (ChebFun toolbox) and Julia (ApproxFun module).*

Keywords: Dispersion curves, Rootfinding, Interval arithmetic, Chebyshev interpolation.

1. Interval arithmetic

Wave propagation in thick plates is well solvable problem. One of part of task solution is to find the waveform dispersion curves. This problem has been chosen for the first trial using interval arithmetic (Moore, Kearfott & Cloud, 2009), because of its relative simplicity. In calculating the dispersion curves is needed to quantify the only trigonometric functions, hyperbolic functions, and square roots. All of these functions are already included in INTLAB (Rump, 1999) and therefore need not to be newly programmed.

The thick plate is defined as that it has a nonzero thickness 2*d* and endless remaining dimensions. To calculate the stress wave propagation in plates it is used the integration along the dispersion curves for thick plates. These dispersion relations are (for the stress-free boundary conditions of the plate surfaces and for symmetric modes) defined as

$$(\xi^2-2)^2\tanh(\gamma d\sqrt{1-\xi^2}) - 4\cdot\sqrt{1-\xi^2}\sqrt{1-\kappa\xi^2}\tanh(\gamma d\sqrt{1-\kappa\xi^2}) = 0,$$

where γ is wavenumber ($\gamma = 2\pi/\lambda$), ξ is the ratio of the phase velocity and the shear wave velocity and κ mean the ratio of the squares of the phase velocities for the plate's material.

To avoid computation in the complex domain it was necessary to break up calculation into the parts according to parameter ξ value (see Tab. 1) as the algorithm was implemented in INTLAB. For each interval are the equations shown in tab 1. To quantify the equations and finding the waveform dispersion curves is used interval arithmetic.

Tab. 1: Dispersion equation: $(\xi^2-2)^2\cdot\alpha - 4\cdot\beta\cdot x_1 x_2 = 0$.

	$0<\xi<1$	$1<\xi<1/\sqrt{\kappa}$	$1/\sqrt{\kappa}<\xi$
x_1	$\sqrt{1-\xi^2}$	$\sqrt{\xi^2-1}$	$\sqrt{\xi^2-1}$
x_2	$\sqrt{1-\kappa\xi^2}$	$\sqrt{1-\kappa\xi^2}$	$\sqrt{\kappa\xi^2-1}$
α	$sinh(\gamma d x_1)\,cosh(\gamma d x_2)$	$sin(\gamma d x_1)\,cosh(\gamma d x_2)$	$sin(\gamma d x_1)\,cos(\gamma d x_2)$
β	$cosh(\gamma d x_1)\,sinh(\gamma d x_2)$	$cos(\gamma d x_1)\,sinh(\gamma d x_2)$	$-\,cos(\gamma d x_1)\,sin(\gamma d x_2)$

Interval arithemic is an extension of arithmetic over real numbers, where for each real function $f(x_1, \ldots, x_n)$, the interval function $F(X_1, \ldots, X_n)$ is called an interval extension of the function f if for each

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intervals $I_1, \ldots, I_n$ function $F(I_1, \ldots, I_n)$ returns interval I such that $\forall y_1 \in I_1 \ldots \forall y_n \in I_n(f(y_1, \ldots, y_n) \in I)$. For other applications it is particularly important the interval Newton method, according to which each continuously differentiable function f each interval I must be $\forall a, x \in I, \exists \varepsilon \in I(f(x) = f(a)+(x-a)f'(\varepsilon)$. Specifically then, the function f continuously differentiable on the interval I has all the roots in I in interval $N_a = a-(F(a)/F'(I))$ where a is an arbitrary element of I and F, F' is an interval extension function of f, f'.

For quantification of interval arithmetic was used MATLAB's toolbox INTLAB, which are defined not only the basic functions for interval arithmetic, but their automatic differentiation too (Bücker, Corliss, Hovland, Naumann, & Norris, 2005).

When trying to use the Newton method in INTLAB was needed to solve the problem by dividing intervals. INTLAB always returns the result $\langle -\infty, \infty \rangle$ for each I/J where $0 \in J$. However, for this case, it was necessary defined alternative way of dividing, when the interval is divided into two portions, and finding roots thus diverges into two tasks. The actual calculation is solved in recursive steps. In a single step in the equation $N_a = a-(F(a)/F'(I))$ is substituted middle I per a, yielding a new interval $J = I \cap N_a$. This one is used in the next recursive step. The calculation continues until the result interval width falls under a predetermined accuracy.

2. Chebyshev interpolation

To calculate the stress wave propagation for a longitudinal impact of semi-infinite thick cylindrical bars it is used the integration along the dispersion curves. These dispersion relations $f(\xi, \gamma a)$ is defined as

$$(2-\xi^2)^2 J_0(\gamma a\, A)\, J_1(\gamma a\, B) + 4AB\, J_1(\gamma a\, A)\, J_0(\gamma a\, B) - \frac{2\xi^2}{\gamma a} A\, J_1(\gamma a\, A)\, J_1(\gamma a\, B) = 0,$$

where a is radius of the semi-infinite bar, γ is wavenumber, ξ is the ratio of the phase velocity and the shear wave velocity, κ means the ratio of the squares of the phase velocities for the bar's material, $A = \sqrt{\kappa\xi^2 - 1}$, $B = \sqrt{\xi^2 - 1}$ and J is the Bessel function of the first kind.

Summary of the Chebyshev expansion algorithm (Boyd, 1995):

1. Choose the following:
   1. γa
   2. Search interval, $\xi \in [a, b]$.

      The search interval must be chosen by physical and mathematical analysis of the individual problem. The choice of the search interval $[a, b]$ depends on the user's knowledge of the physics of his/her problem, and no general rules are possible.
   3. The number of grid points, N.

      N may be chosen by setting $N = 1+2^m$ and the increasing N until the Chebyshev series displays satisfactory convergence. To determine when N is sufficiently high, we can examine the Chebyshev coefficients a_j, which decrease exponentially fast with j.
2. Compute a Chebyshev series, including terms up to and including T_N, on the interval $\xi \in [a, b]$.

   1. Create the interpolation points (Lobatto grid):

      $$\xi_k \equiv \frac{b-a}{2}\cos\left(\pi\frac{k}{N}\right) + \frac{b+a}{2}, \qquad k = 0,1,2, \ldots, N.$$
   2. Compute the elements of the $(N+1)\times(N+1)$ interpolation matrix.

      Define $p_j = 2$ if $j = 0$ or $j = N$ and $p_j = 1$, $j \in [1, N-1]$. Then the elements of the interpolation matrix are

$$I_{jk} = \frac{2}{p_j p_k N} \cos\left(j\pi \frac{k}{N}\right).$$

3. Compute the grid-point values of $f(\xi)$, the function to be approximated:

$$f_k \equiv f(\xi_k), k = 0,1, \dots, N.$$

4. Compute the coefficients through a vector-matrix multiply:

$$a_j = \sum_{k=0}^{N} I_{jk} f_k \,, \qquad j = 0, 1, 2, \dots, N.$$

The approximation is

$$f_k \approx \sum_{j=0}^{N} a_j T_j \left(\frac{2\xi - (b+a)}{b-a}\right) = \sum_{j=0}^{N} a_j \cos\left\{ j \cos^{-1}\left(\frac{2\xi - (b+a)}{b-a}\right)\right\}.$$

3. Compute the roots of f_N as eigenvalues of the Chebyshev–Frobenius matrix

   Frobenius showed that the roots of a polynomial in monomial form are also the eigenvalues of the matrix which is now called the *Frobenius companion matrix*. Day and Romero (2005) developed a general formalism for deriving the *Frobenius matrix* for any set of orthogonal polynomials.

4. Refine the roots by Newton iteration with $f(\xi)$ itself.

   Once a good approximation to a root is known, it is common to *polish* the root to close to machine precision by one or two Newton iterations.

Computations were performed with the normalized Bessel functions that eliminate the large fluctuations in magnitude. For numerical experiments were used MATLAB’s toolbox CHEBFUN (Driscoll, Hale & Trefethen, 2014) and Julia’s package ApproxFun (ApproxFun).

3. Conclusions

Fig. 1 shows the dispersion curves as calculated using a classic root-finding algorithm and the Chebyshev method. It can be seen, that both approaches provide equivalent results.

Both the interval arithmetic and the Chebyshev interpolation provide a robust method for finding the roots of the dispersion equation.

For relatively low speed, these methods are not suitable to complete the calculation, it is however possible to use the first approach with low accuracy, or when using the Gaussian integration method along the dispersion curves.

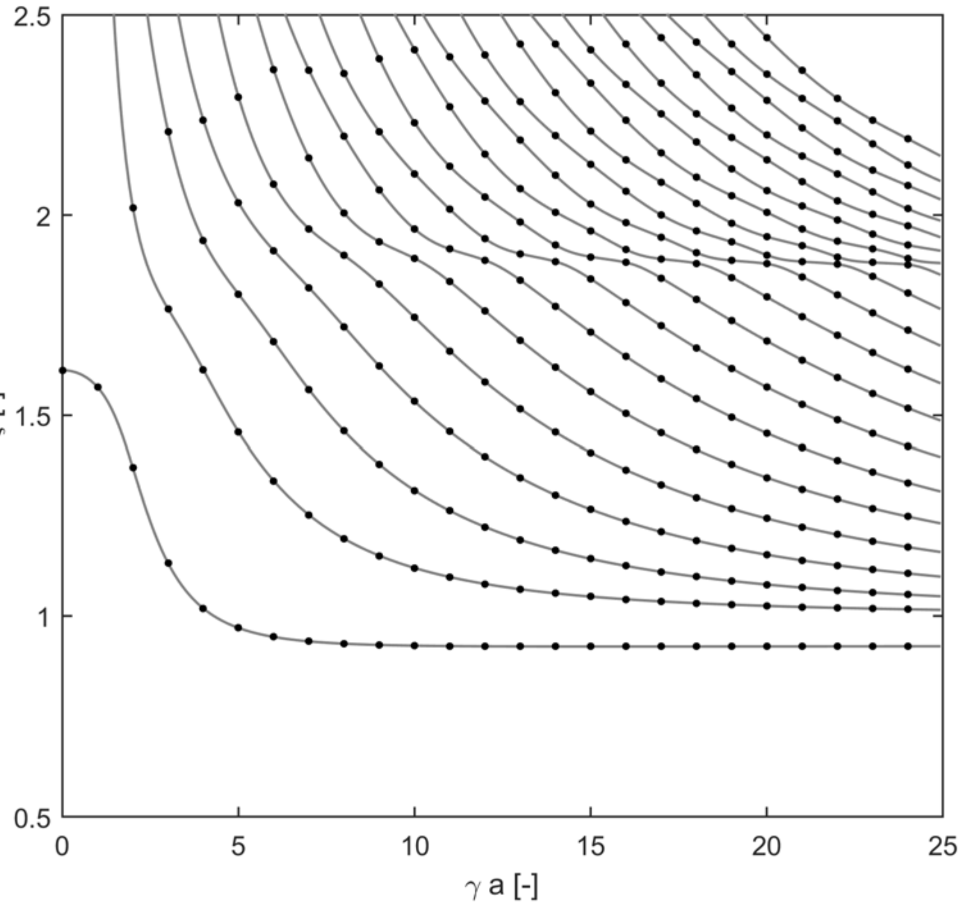

Fig. 1: Dispersion curves computed using a conventional mode tracing algorithm (gray lines) and the Chebyshev method (black dots).

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URL: github.com/ApproxFun/ApproxFun.jl.git