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THEORY, EXPERIMENT AND NUMERICAL APPROACH FOR THE BEAM RESTED ON NONLINEAR ELASTIC FOUNDATION

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Abstract: *This work presents theory, approximations of experiments and numerical approaches suitable for the solution of straight plane beams rested on an elastic (Winkler's) foundation. The nonlinear dependence of the reaction force can be described via bilateral tangent-linear or secant-linear or nonlinear (linear + arcus tangent) approximations. These applications lead to linear or nonlinear differential equations of 4th-order (Central Difference Method, Newton-Raphson Method).*

Keywords: Elastic foundation, Beam, Central Difference, Nonlinearity, Experiment.

1. Introduction

Beams on elastic foundations are frequently used in the practice. In mechanics, the beam is described by differential equation $\frac{d^4v}{dx^4} - \frac{\mathrm{q_R}}{EJ_{ZT}} = 0$, where $v = v(x)$ [m] is deflection of the beam, E [Pa] is the modulus of elasticity of the beam, J_{ZT} [m^4] is the major principal second moment of the beam cross-section and $\mathrm{q_R} = \mathrm{q_R}(x, v, \ldots)$ [Nm^{-1}] is the nonlinear reaction force in the foundation (Frydrýšek et al. 2013; Frydrýšek et al. 2014). Our work focuses on the solution of straight 2D beams on an elastic foundation with nonlinear behaviour (evaluation of experiments, curve fitting). For typical cases, the linear Bernouli's beam theory is coherent with nonlinear response of reaction force in foundation.

2. Evaluation of Experiment

The methodology for measuring of elastic foundation applied in this paper is based on the pressing of a beam into the foundation; see Fig. 1, Table 1 and reference (Klučka et al., 2014)) (i.e. dependence $\mathrm{q_R} = \mathrm{q_R}(v)$ which is based on the foundation load-settlement behaviour is evaluated and approximated).

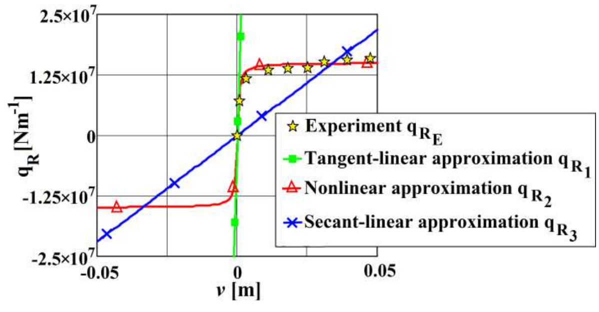

Fig. 1: Dependence of reaction force on deflection (i.e. foundation load-settlement behavior for a sand) - experiment and its suitable linear and nonlinear approximations.

The measured nonlinear behaviour of reaction force on displacement in the foundation (i.e. dependence $\mathrm{q_R} = \mathrm{q_{R_E}}$) can be approximated by bilateral tangent-linear $\mathrm{q_{R_1}}$ or bilateral nonlinear $\mathrm{q_{R_2}}$ or bilateral secant-linear $\mathrm{q_{R_3}}$ functions; see Table 1 and Fig. 1. From Fig. 1, in comparing with

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experiment, the best curve fitting is performed via nonlinear behaviour of foundation prescribed by function $q_{R_2} = k_1 v + k_a \mathrm{arctg}(c_a v)$.

Tab. 1: Approximations for reaction forces in the foundation.

Description	Constants:
Tangent-linear approx.: $q_{R_1} = k_1 v$, linear diff. equation $\frac{d^4 v}{dx^4} - \frac{k_1 v}{EJ_{ZT}} = 0$. Good fitting for small values of *v*.	k_1=1.7422×10^{10} Nm^{-2}, k_a=0 Nm^{-2}, c_a=0 m^{-1}.
Nonlinear approx.: $q_{R_2} = k_1 v + k_a \mathrm{arctg}(c_a v)$, nonlinear diff. equation $\frac{d^4 v}{dx^4} - \frac{k_1 v + k_a \mathrm{arctg}(c_a v)}{EJ_{ZT}} = 0$. Good fitting for all values of *v* (i.e. the closest to the experiment).	k_1=5.21×10^5 Nm^{-2}, k_a=9.52×10^6 Nm^{-2}, c_a=1.83×10^3 m^{-1}.
Secant-linear approx.: $q_{R_3} = k_1 v$, linear diff. equation $\frac{d^4 v}{dx^4} - \frac{k_1 v}{EJ_{ZT}} = 0$. Good fitting for bigger values of *v*.	k_1=4.3866×10^8 Nm^{-2}, k_a=0 Nm^{-2}, c_a=0 m^{-1}.

3. Central Differences

The FEM is frequently used for solutions of nonlinear problems. However, in this article, the central differences (CD) are applied for their easy derivation of problem. The CD proceed by replacing the derivatives $v_i^{(1)}$, $v_i^{(2)}$, $v_i^{(3)}$ and $v_i^{(4)}$ in the differential equations at the point "i" with step $\Delta = \frac{L}{n}$ [m], where L [m] is length and n [1] is number of divisions. Hence, $v_i^{(1)} = \frac{dv}{dx} \approx \frac{v_{i+1} - v_{i-1}}{2\Delta}$, $v_i^{(2)} = \frac{d^2 v}{dx^2} \approx \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2}$, $v_i^{(3)} = \frac{d^3 v}{dx^3} \approx \frac{v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2}}{2\Delta^3}$, $v_i^{(4)} = \frac{d^4 v}{dx^4} \approx \frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{\Delta^4}$, see Jones, 1997 and Frydrýšek et al., 2014.

4. Solved Example and its Boundary Conditions

The beam of length 2L with cross-section b×h=0.2×0.4 m^2 is resting on an elastic foundation and loaded by force F = 7×10^6 N; see Fig. 2. Beam properties are *E* = 2×10^{11} Pa, $J_{ZT} = \frac{bh^3}{12} = \frac{0.2 \times 0.4^3}{12}$ m^4. Foundation properties are described as an evaluation of experiment in Tab. 1. The beam is symmetrical (i.e. it is sufficient to solve ½ of the beam $x \epsilon \langle 0; L \rangle$). Hence, the beam is described by the equation

$$\frac{d^4 v}{dx^4} - \frac{k_1 v + k_a arctg(c_a v)}{EJ_{ZT}} = 0. \quad (1)$$

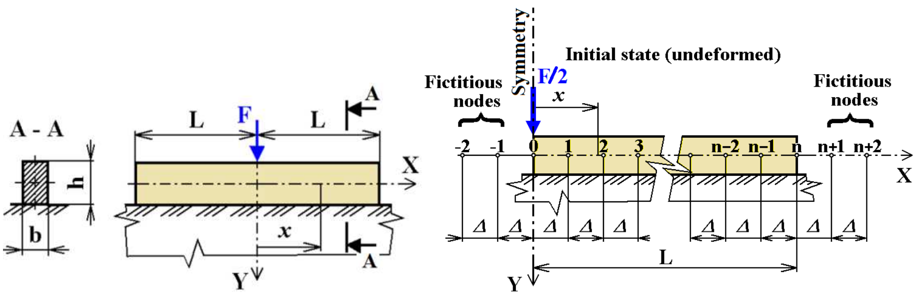

Fig. 2: Beam of length 2L resting on an elastic foundation and loaded by force F and divisions of the beam (CDM – one half of the beam).

From the boundary conditions prescribed in points *x* = 0 m and *x* = L follow equations

$$\left.\begin{array}{c} \frac{dv(x=0)}{dx} = 0, \\ T(x=0) = -\frac{F}{2} => -EJ_{ZT}\frac{d^3 v(x=0)}{dx^3} = -\frac{F}{2} => \frac{d^3 v(x=0)}{dx^3} = \frac{F}{2EJ_{ZT}}, \\ M_o(x=L) = 0 => -EJ_{ZT}\frac{d^2 v(x=L)}{dx^2} = 0 => \frac{d^2 v(x=L)}{dx^2} = 0, \\ T(x=L) = 0 => -EJ_{ZT}\frac{d^3 v(x=L)}{dx^3} = 0 => \frac{d^3 v(x=L)}{dx^3} = 0, \end{array}\right\} \quad (2)$$

where $T(x)$ [N] is shearing force and $M_o(x)$ [Nm] is bending moment.

5. Solved Example and Central Difference Method (CDM)

The beam and its surroundings can be divided into n+5 nodes "i"; see Fig. 2. Denote for simplicity $b = \frac{F\Delta^3}{EJ_{ZT}}$, $a_1 = \frac{k_1\Delta^4}{EJ_{ZT}}$, $a_2 = \frac{k_a\Delta^4}{EJ_{ZT}}$ and $c = 6 + a_1$. Boundary conditions (2) can be approximated by CD, for node "0" (i.e. $\mathrm{i} = 0, x = 0$) and for node "n" (i.e. $\mathrm{i} = \mathrm{n}, x = \mathrm{L}$) as

$$\left.\begin{aligned} \frac{v_{\mathrm{i}+1}-v_{\mathrm{i}-1}}{2\Delta} &= 0 \Rightarrow v_1 - v_{-1} = 0, \\ \frac{v_{\mathrm{i}+2}-2v_{\mathrm{i}+1}+2v_{\mathrm{i}-1}-v_{\mathrm{i}-2}}{2\Delta^3} = \frac{\mathrm{F}}{2EJ_{ZT}} &\Rightarrow v_2 - 2v_1 + 2v_{-1} - v_{-2} = b,, \\ \frac{v_{\mathrm{i}+1}-2v_{\mathrm{i}}+v_{\mathrm{i}-1}}{\Delta^2} &= 0 \Rightarrow v_{\mathrm{n}+1} - 2v_{\mathrm{n}} + v_{\mathrm{n}-1} = 0, \\ \frac{v_{\mathrm{i}+2}-2v_{\mathrm{i}+1}+2v_{\mathrm{i}-1}-v_{\mathrm{i}-2}}{2\Delta^3} &= 0 \Rightarrow v_{\mathrm{n}+2} - 2v_{\mathrm{n}+1} + 2v_{\mathrm{n}-1} - v_{\mathrm{n}-2} = 0. \end{aligned}\right\} \tag{3}$$

Similarly, differential equation (1) can be approximated via CD for q_{R_1}, q_{R_2} and q_{R_3} as

$$v_{\mathrm{i}-2} - 4v_{\mathrm{i}-1} + (6 + a_1)v_{\mathrm{i}} - 4v_{\mathrm{i}+1} + v_{\mathrm{i}+2} + a_2\mathrm{arctg}(c_a v_{\mathrm{i}}) = 0, \quad \mathrm{i} = 0,2, \dots, \mathrm{n}, \tag{4}$$

Now, the variables v_{-2}, v_{-1}, $v_{\mathrm{n}+1}$ and $v_{\mathrm{n}+2}$ (i.e. results in fictitious nodes -2, -1, n+1 and n+2, see Fig. 2) can be expressed from boundary conditions (3). Hence the set of nonlinear equations can be written in the matrix form as

$$f_i = [\mathrm{M}]\{v\} + a_2\mathrm{arctg}(c_a\{v\}) - \{\mathrm{b}\} = 0, \tag{5}$$

where

$$[\mathrm{M}] = \begin{bmatrix} c & -8 & 2 & 0 & 0 & 0 & 0 & \dots & 0 \\ -4 & 7+a_1 & -4 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -4 & c & -4 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -4 & c & -4 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -4 & c & -4 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -4 & c & -4 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & -4 & 5+a_1 & -2 \\ 0 & \dots & 0 & 0 & 0 & 0 & 2 & -4 & 2+a_1 \end{bmatrix}, \{b\} = \begin{Bmatrix} b \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \{v\} = \begin{Bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{Bmatrix} \tag{6}$$

6. Iterative Approach and Solutions

The system of coupled nonlinear equations can be solved iteratively via Newton–Raphson Method as

$$\left\{{}^{\langle \mathrm{j}+1\rangle}v\right\} = \left\{{}^{\langle \mathrm{j}\rangle}v\right\} - \left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right]^{-1}\left\{[\mathrm{M}]\left\{{}^{\langle \mathrm{j}\rangle}v\right\} + a_2\mathrm{arctg}\left(c_a\left\{{}^{\langle \mathrm{j}\rangle}v\right\}\right) - \{\mathrm{b}\}\right\}, \tag{7}$$

where vectors $\left\{{}^{\langle \mathrm{j}+1\rangle}v\right\}$ and $\left\{{}^{\langle \mathrm{j}\rangle}v\right\}$ are new and old iterations and matrix $\left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right] = \left(\frac{\partial f_{\mathrm{i}}}{\partial\, {}^{\langle \mathrm{j}\rangle}v_{\mathrm{k}}}\right)_{\mathrm{i,k}=0,1,2,\dots,\mathrm{n}}$ is the Jacobian matrix. Some results are presented in Fig. 3.

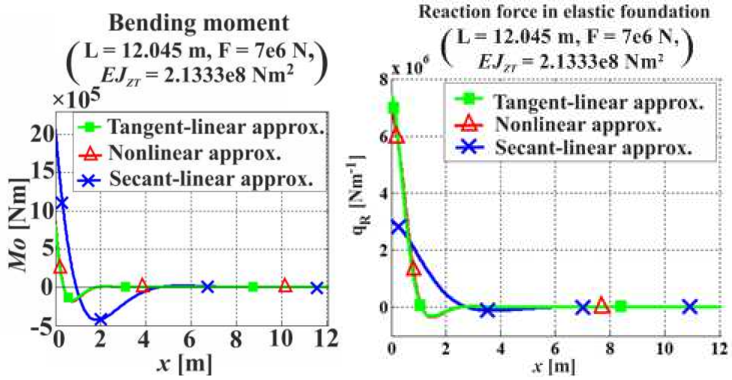

Fig. 3. Dependence for bending moment and distributed reaction forces on coordinate x of the beam for different types of foundation approximations.

The differences between the linear and nonlinear approximations are evident.

7. Conclusions

The use of an elastic foundation including nonlinearities is a suitable way of performing numerical/experimental modelling of engineering problems in the branch of mechanics and biomechanics; see some applications in references (Frydrýšek et al., 2013). Our team is in the process of application of elastic foundation as a suitable simplification of the complicated interaction between implants and bones in traumatology and orthopaedics, see Fig. 4.

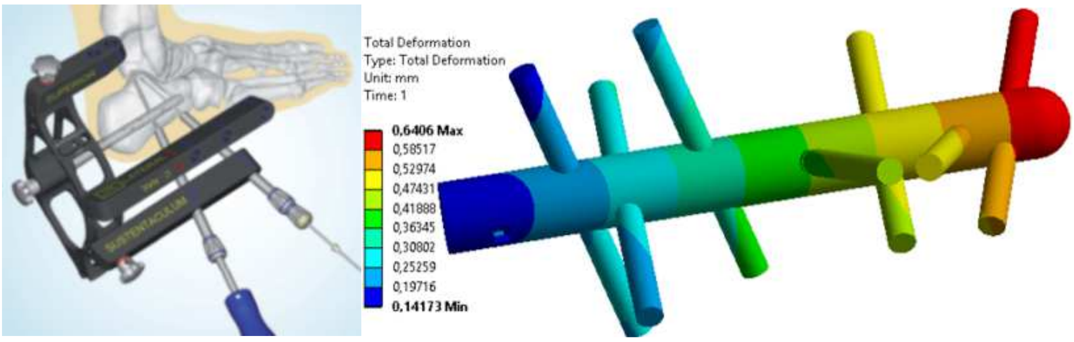

Fig. 4. The intramedullary nail C-NAIL for minimal-invasive fixation of intraarticular calcaneal fractures (application of FEM and evaluation as structures on elastic foundation).

The derivation, rapid solutions and application of our own simple numerical model based on the Central Difference Method (CDM) open up a new avenue for further applications using a stochastic approach (i.e. millions of solutions with random inputs). The Simulation-Based Reliability Assessment Method (i.e. the direct Monte Carlo approach) can be applied. For more information see (Frydrýšek et al., 2013; Marek et al., 1995).

This work is a continuation of our previous work. The measured material properties of the elastic foundation were evaluated and approximated in three ways (via bilateral tangent-linear, bilateral nonlinear and secant-linear functions). Beams on elastic linear/nonlinear foundations were solved via CDM and Newton's (Newton-Raphson) Method.

Theory, experiment, CDM and numerical approach are exposed for the beams rested on linear/nonlinear foundations. The best approximation is the bilateral nonlinear function $\mathrm{q}_{\mathrm{R}_2} = k_1 v + k_a \mathrm{arctg}(c_a v)$.

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