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CONCRETE PLASTICITY MODEL AND ITS APPLICATION TO PLANE STRESS STATE

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Abstract: *Two Drucker-Prager criteria are employed to define a nonlinear material model for concrete that is capable to solve tri-axial analysis of plasticity. The model consists of one Drucker-Prager criterion set to approximate tensile stress area and the other to describe compression area. The first derivative singularities are treated by using an imaginary tangent as a local yield function and the model is modified for specific application for the plane stress state. The model is implemented into the SIFEL solver using the finite element method.*

Keywords: Double Drucker-Prager, Concrete plasticity model, Plane stress state, SIFEL solver, Finite element method.

1. Introduction

Since behaviour of concrete in tension and compression is diverse, single yield criterion for this material is not sufficient enough to describe both stress areas. In this case, two Drucker-Prager yield criteria can be employed to form a concrete plasticity model capable to capture more general stress-strain states. Combining these two criteria nevertheless leads to the first derivative singularities which need to be treated separately. Compared to other concrete models, e.g. Drucker-Prager + Tresca (Feenstra & de Borst, 1996), this plasticity model incorporates less singularities. Moreover, whole plasticity calculation of the model can be performed at the level of stress invariants and the treatment of the singularities can be thus considerably simplified. The model is also modified for the plane stress state where the correction of the out-of-plane elastic strain is needed. The model has been implemented into the SIFEL software package (Krejčí, Koudelka & Kruis, 2011) using the finite element method.

In the first part of the paper, the principle of general plasticity and the single Drucker-Prager criterion is described. The second part is dedicated to the treatment of the singularities and especially to the out-of-plane strain correction regarding the plane stress state. In the end, an example of calculation using the model is presented where the correction for the plane stress state should be demonstrated.

2. Plasticity in finite element method

The state, in which material exhibits plastic flow, is generally expressed by the following yield criterion

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0 \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress vector, $\boldsymbol{q}$ is the vector of internal variables and f is the yield function. If values of the yield function are negative $f < 0$, material is located in the elastic stress area. Stress states where the yield function provides positive values, are not admissible.

Assuming elastoplastic behaviour and small strains, relation between stress and strain can be determined by

$$\boldsymbol{\sigma} = \boldsymbol{D}_e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) \tag{2}$$

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where $\boldsymbol{\varepsilon}$ is total strain vector, $\boldsymbol{\varepsilon}_p$ is plastic strain vector and $\boldsymbol{D}_e$ denotes elastic stiffness matrix. In the case of the concrete plasticity model, the associated flow rule is selected as an expression of plasticity

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\gamma}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{3}$$

where $\dot{\boldsymbol{\varepsilon}}_p$ represents the rate of plastic strains, $\dot{\gamma}$ is the rate of plastic multiplier that indicates the magnitude of plastic strains, and the gradient of the yield function dictates the direction of plastic flow.

A nonlinear analysis is usually carried out by an iterative calculation with increments of load. As the first step, the evaluation of the trial stress state $\boldsymbol{\sigma}_{tr}$ is performed using the following expression

$$\boldsymbol{\sigma}_{tr} = \boldsymbol{D}_e\left(\boldsymbol{\varepsilon}^{(n)} - \boldsymbol{\varepsilon}_p^{(n-1)}\right) \tag{4}$$

where the superscript n indicates the iteration step of the global iterative procedure. The trial stress state that is calculated from total strains of the current step and plastic strains of the previous step, is determined whether it belongs to the admissible stress area. If it does, no further computation of plasticity is performed. Otherwise, the trial stress state is necessary to return to the admissible stress area by, for example, the cutting plane method (de Souza Neto, Perić & Owen, 2008).

Assuming incremental calculation, eq. (3) can be rewritten as

$$\Delta\boldsymbol{\varepsilon}_p^{(n)} = \Delta\gamma^{(n)}\frac{\partial f^{(n-1)}}{\partial \boldsymbol{\sigma}} \tag{5}$$

The cutting plane method is specifically used for determination of the increment of the plastic multiplier $\Delta\gamma$. With the help of the newly calculated increment of the plastic strain vector using eq. (5), the corrected stress state is then possible to evaluate

$$\boldsymbol{\sigma}^{(n)} = \boldsymbol{\sigma}^{tr} - \boldsymbol{D}_e\Delta\gamma^{(n)}\frac{\partial f^{(n-1)}}{\partial \boldsymbol{\sigma}} \tag{6}$$

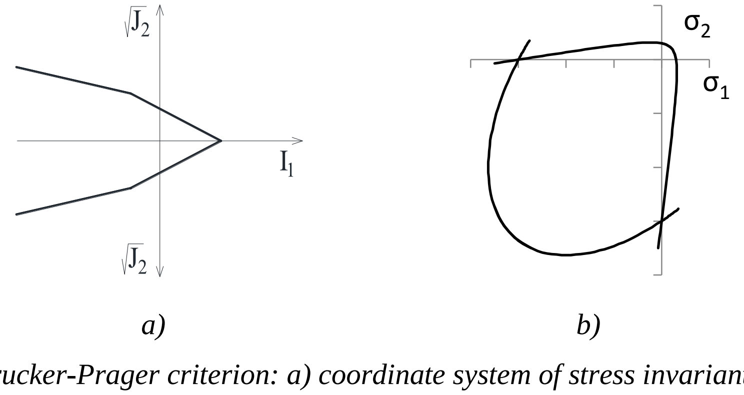
3. Double Drucker-Prager concrete plasticity model

The single Drucker-Prager plasticity is described by the following yield function

$$f(\boldsymbol{\sigma}) = \alpha_\phi I_1(\boldsymbol{\sigma}) + \sqrt{J_2(\boldsymbol{\sigma})} - \tau_0 \tag{7}$$

where τ_0 is the parameter representing shear strength and the parameter α_ϕ is connected with friction angle. I_1 is the first invariant of the stress tensor and J_2 is the second invariant of the deviatoric stress tensor. Exploiting of these two yield functions with different setup of the parameters leads to a suitable plasticity model that describes concrete in both compression and tension (Jirásek & Bažant, 2002). By employing concrete strength in single compression f_c and in biaxial compression f_b, parameters for yield function that approximates compressive behavior, can be calculated

$$\alpha_{\phi,c} = \frac{\sqrt{3}}{3}\frac{f_b - f_c}{2f_b - f_c}, \quad \tau_{0,c} = f_c\frac{\sqrt{3} - 3\alpha_{\phi,c}}{3}. \tag{8}$$



a) b)

Fig. 1: Double Drucker-Prager criterion: a) coordinate system of stress invariants, b) plane stress state

The tensile stress area can be then described by adding concrete strength in tension f_t

$$\alpha_{\phi,t} = \frac{\sqrt{3}}{3} \frac{f_c - f_t}{f_c + f_t}, \quad \tau_{0,t} = f_c \frac{\sqrt{3} - 3\alpha_{\phi,t}}{3}. \tag{9}$$

The resulting yield surface, created by two separate Ducker-Prager yield criteria, is represented by an angular cone whose characteristic sections are displayed in fig. 1.

4. Singularities treatment

As depicted in fig. 1a, the first derivative singularities are located at the vertex of the cone and at the intersection of the criteria. By these singularities, so called stress return areas are established where the value and the derivative of the relevant yield functions are evaluated. The whole problem is solved at the level of the stress invariants and the derivative of the yield functions are substituted by

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{df}{dJ_2} \cdot \frac{\partial J_2}{\partial \boldsymbol{\sigma}} + \frac{df}{dI_1} \cdot \frac{\partial I_1}{\partial \boldsymbol{\sigma}}. \tag{10}$$

Return to the point of the first derivative singularity is solved by using an imaginary tangent which is defined as a normal to the connecting line between the singularity and the trial stress state point. The tangent serves as a local yield function and its value together with its derivative is used further in calculation, specifically in the cutting plane method.

5. Correction for plane stress state

The plane stress state is constrained by the following expressions

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad \gamma_{xz} = \gamma_{yz} = 0. \tag{11}$$

Elastic stiffness matrix for the plane stress state is derived from general Hooke's law in a way that it is assumed that, apart from other conditions, the component σ_z is in elastic form equal to zero. However, if a tri-axial plasticity model is used, a stress return algorithm usually generates in the case of the plane stress state non-zero out-of-plane stress component σ_z which may negatively influence whole plasticity calculation. Stress correcting procedure that deals with this problem, has been firstly suggested by (Aravas, 1987) and the following algorithm, used in SIFEL, has been designed by (Dodds, 1987).

The non-zero stress components together with corresponding strains are assembled as follows

$$\boldsymbol{\sigma} = \{\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}\}^T, \quad \boldsymbol{\varepsilon} = \{\varepsilon_{11} \quad \varepsilon_{22} \quad 2\varepsilon_{12}\}. \tag{12}$$

In the first step of the procedure, a trial out-of-plane strain component is calculated

$$\varepsilon_{33}^{e\,trial} = -\frac{\nu}{1-\nu}\left(\varepsilon_{11}^{e} + \varepsilon_{22}^{e}\right). \tag{13}$$

As the next step, a stress return algorithm is called while using an extended stiffness matrix (as for axisymmetric problem) for calculating trial stresses

$$\begin{bmatrix} \boldsymbol{\sigma}^{trial} \\ \hline \sigma_{33}^{trial} \end{bmatrix} = \boldsymbol{D}_{ext} \begin{bmatrix} \boldsymbol{\varepsilon}^{e} \\ \hline \varepsilon_{33}^{e\,trial} \end{bmatrix}. \tag{14}$$

After the stress return algorithm is finished, the stress state $[\boldsymbol{\sigma}^T \quad \sigma_{33}]^T$ that satisfies a plasticity criterion, is obtained. The stress correcting algorithm is terminated if the out-of-plane stress component σ_{33} is lesser than tolerable inaccuracy. Otherwise, the following correction of the out-of-plane strain is applied

$$\varepsilon_{33}^{e} = \varepsilon_{33}^{e\,trial} - \frac{\sigma_{33}}{D_{44}} \tag{15}$$

where D_{44} is the element of the extended stiffness matrix at the 4,4 position. The newly acquired out-of-plane strain then enters to the stress return algorithm and again the value of the out-of-plane stress component is tested. This procedure continues until the required value of the out-of-plane stress is reached. The general iterative form of eq. (15) can be then written as

$$\varepsilon_{33}^{i+1} = \varepsilon_{33}^{i} - \frac{\sigma_{33}^{i}}{D_{44}} . \tag{16}$$

6. Calculation example

The plasticity model is tested on the example of a concrete beam with following parameters: 0.6 m height, 3.6 m length, 0.3 m depth, concrete C30/37 f_c = 30 MPa, f_t =3 MPa, f_b = 1.2 f_c. The beam is fixed alongside the left edge and loaded by vertical force at the upper right corner. The results are shown for the 33,5 kN value of the loading force. In accordance to the development of the plastic multiplier (fig. 2), the distribution of the out-of-plane stress component can be observed from fig. 3. In the areas where behaviour of concrete is elastic, the out-of-plane stress is automatically equal to zero. However, in the case of the plastic yielding area, minor values that corresponds with tolerable inaccuracy (1 kPa), can be recognized at the upper left edge. These small values in the plastic yielding area prove the presence of the stress correcting algorithm inside the plasticity model.

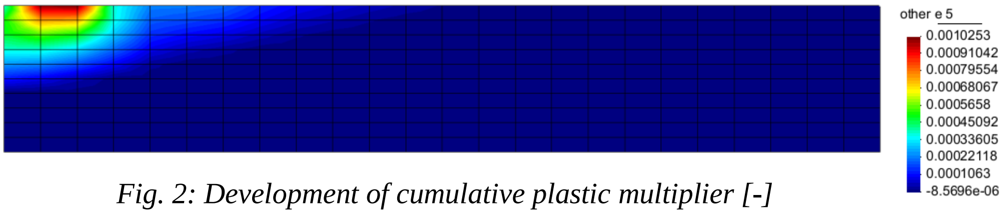


Fig. 2: Development of cumulative plastic multiplier [-]

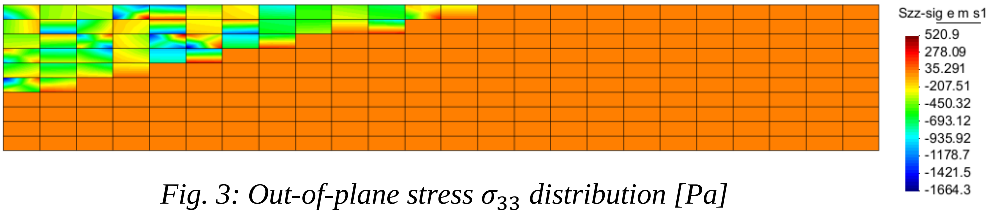


Fig. 3: Out-of-plane stress σ_{33} distribution [Pa]

7. Conclusions

The concrete plasticity model using two Drucker-Prager criteria has been presented in the paper while emphasizing the solution of the first derivative singularities and the importance of the stress correcting algorithm in case of the plane stress state. The validity of the algorithm has been demonstrated in the plastic yielding area of the concrete beam.

Acknowledgement

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