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ON THE ISOGEOMETRIC FORMULATION OF PLANAR CURVED BEAMS

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Abstract: *An isogeometric formulation of planar Timoshenko beams with variable curvature is presented. The Non-Uniform Rational B-Splines (NURBS) are used for both the geometry and unknown approximations. The NURBS capability of exact geometry representation, which is independent of a mesh density, is very advantageous in the analysis of curved beams as there is no loss of accuracy caused by the geometry approximation. The high accuracy results can be obtained while keeping the computational cost low. A beam element for arbitrarily curved planar beams is implemented and its performance is verified by means of few simple tests. The problem of shear locking is observed and is overcome by using reduced integration.*

Keywords: Curved Beams, Finite element analysis, Isogeometric analysis, NURBS, Shear locking.

1. Introduction

The finite element method is undoubtedly the most powerful tool in nowadays structural analysis. Its popularity has grown over past few decades and at the moment we can hardly imagine another approach which would substitute the need of finite element analysis completely. A large effort has been made to improve convergence and accuracy of the results, however most of the researchers concentrate only on the analysis itself. If we want to make the design and the analysis really efficient it is necessary to look at the design from the very beginning.

The geometry used for the analysis is usually represented by CAD files. The biggest drawback of such a representation is the need for its discretization prior the analysis. In this phase the geometry approximation and meshing take a turn. This process usually cannot be made fully automatic and it requires about 80% of overall analysis time. Moreover, the loss of the exact geometry can rapidly reduce the accuracy. The gap between CAD model and the finite element analysis can be eliminated by the concept proposed by Hughes et al. (2005) which is referred to as isogeometric analysis.

In isogeometric analysis the same representation of geometry is shared between CAD and FEM models. The exact geometry representation appears to be a big advantage especially for curved geometries such as curved beams and shells. The main emphasis of this paper is on two-dimensional curved beams. In general, curved beams can be modeled using sufficient number of straight beam elements but this approach can lead to the smaller accuracy and the higher computational cost. Among curved beam elements most of the literature is dedicated to the shapes with constant curvature. In this paper the isogeometric formulation of the planar beam element applicable to beams with variable curvature based on work of Bouclier et al. (2012) is presented.

2. NURBS-based curved beam element

The NURBS are generated using B-splines which can be defined recursively using the Cox-de Boor formulas (Cox, 1971, de Boor, 1972). The starting point are the piecewise constant functions

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

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where ξ_i is the coordinate of the i^{th}-knot and parameter $\xi \in (0,1)$ runs through the entire patch (subdomain of knot spans which are seen as “elements” in isogeometric analysis). See Fig. 1 or Piegl and Tiller (1997) for better understanding of NURBS geometry. A pth-degree B-spline function is defined recursively

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \,. \qquad (2)$$

Finally, to generate the NURBS from its non-rational counterparts we use

$$R_i^p(\xi) = \frac{N_{i,p}(\xi) w_i}{\sum_{\hat{i}=1}^{n} N_{\hat{i},p}(\xi) w_{\hat{i}}} \,, \qquad (3)$$

where w_i are the weights associated with the corresponding basis function. The NURBS curve is given by

$$\boldsymbol{C}(\xi) = \sum_{i=1}^{n} R_i^p(\xi) \boldsymbol{P}_i, \qquad (4)$$

where $\boldsymbol{P}_i$ are the Cartesian coordinates of the i^{th}-control point, all control points define a control net of the NURBS curve. The advantage of NURBS over classical polynomial function is the inter-element continuity. While traditional polynomial shape functions provide C^0-continutity between the elements, the NURBS provide up to C^{p-1}-continuity. The lower continuity can be achieved by increasing a knot multiplicity. For the purposes of this paper we restrict ourselves to 2nd order NURBS.

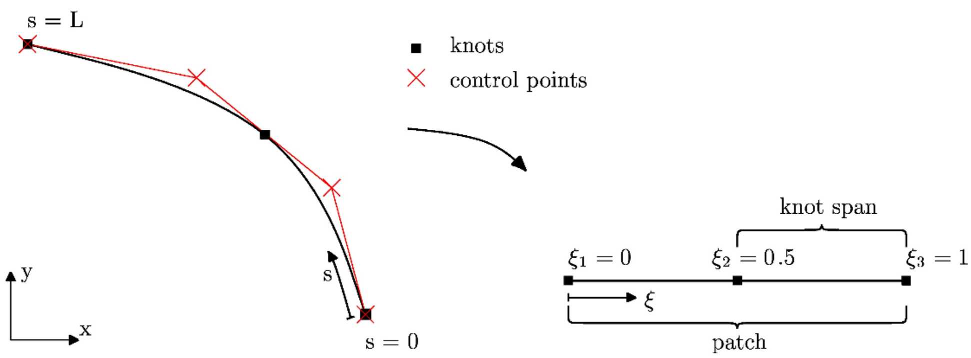

Fig. 1: Description of NURBS finite element geometry.

A curved Timoshenko beam placed in x-y plane is considered. The degrees of freedom are tangential displacement $u_t(s)$, normal displacement $u_n(s)$ and rotation $\theta(s)$. Curvilinear coordinate s runs along the midline of the beam. Membrane, transverse shear and bending strains are given by

$$\varepsilon_m = u_t' - \frac{u_n}{R}, \qquad \gamma_s = \frac{u_t}{R} + u_n' - \theta, \qquad \chi_b = \theta', \qquad (5)$$

where the prime indicates a derivation with respect to the curvilinear coordinate s. Formulas for strain components (5) are used to derive strain-displacement matrix $\boldsymbol{B}$, which is defined as

$$\boldsymbol{\varepsilon} = \boldsymbol{Br}, \qquad (6)$$

where $\boldsymbol{\varepsilon} = \{\varepsilon_m, \gamma_s, \chi_b\}^T$ and $\boldsymbol{r} = \{u_t, u_n, \theta\}^T$. Stiffness matrix is evaluated using

$$\boldsymbol{K} = \int_0^L \boldsymbol{B}^T \boldsymbol{DB} \, ds, \qquad (7)$$

where **D** is a material matrix resulting from

$$N = EA\varepsilon_m, \qquad Q = GA\gamma_s, \qquad M = EI\chi_b, \qquad (8)$$

where N, Q and M are axial force, transverse shear force and bending moment, respectively. Young’s modulus E, shear modulus G, area A and moment of inertia I are the material and cross-section characteristics. To evaluate the stiffness matrix $\boldsymbol{K}$ the Gauss quadrature is used, three point rule is sufficient for the integration.

3. Numerical results

A NURBS beam element has been implemented and its performance has been verified by means of three simple tests (see Fig. 2). Firstly, the analysis of a cantilever beam has been performed and results have been compared with the exact solution. In case of moment loading, the linear distribution of rotation is expected. It has been proven (see Fig. 3) that the right results can be obtained only in case of parametrization which leads to constant Jacobian. For constant Jacobian the non-uniformly distributed control points are needed. When the control points are distributed uniformly, the Jacobian is not constant and the patch test is not satisfied.

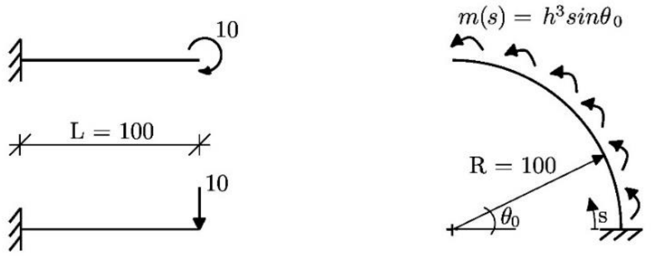

Fig. 2: Cantilever beam subjected to force load and moment load and curved beam with sinusoidal moment loading. ($E = 10^5, I = 0.0833, h = 1.0, b = 1.0, \nu = 0.0$)

In case of a cantilever beam subjected to force load the shear locking phenomena is observed and the expected linear curvature is not obtained (Fig. 4). Selective reduced integration has been used to remove shear locking and thus only two Gauss points have been used for integration of shear components, instead of three Gauss points used for membrane and bending components. This treatment is sufficient to overcome shear locking. Note that Bouclier et al. (2012) have proposed even more efficient quadrature schemes, which use the fact that due to the higher inter-element continuity (in comparison with classical polynomial functions) less Gauss points can be used over entire patch.

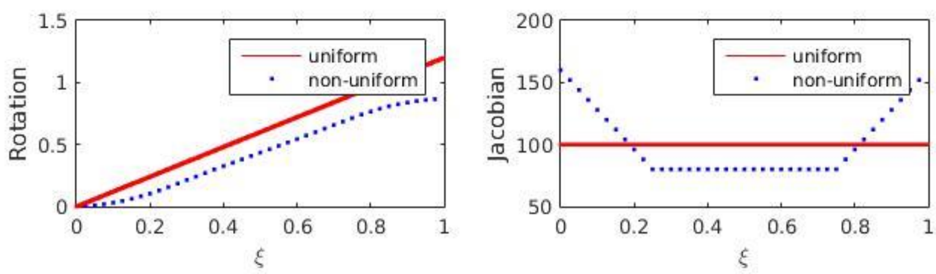

Fig. 3: Cantilever beam (moment load): Rotation and Jacobian corresponding to the linear parametrization (non-uniformly distributed control points) and the non-linear parametrization (uniformly distributed control points).

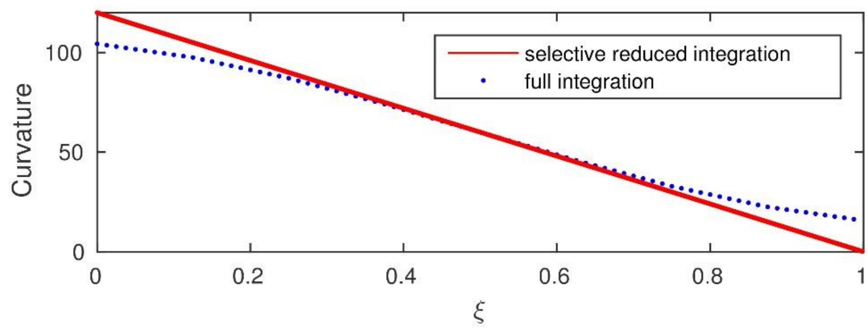

Fig. 4: Cantilever beam (force load): Selective reduced integration involving two Gauss points sufficiently removes shear locking and linear curvature is obtained.

Finally, a curved beam subjected to the sinusoidal loading has been analyzed. In Fig. 5a) the convergence of NURBS elements is compared to the classical polynomial straight beam elements implemented in the OOFEM finite element code (Patzák, 2014). With the NURBS element we can obtain almost the exact solution for small number of nodes while the high number of elements would be needed when traditional straight elements are used. In Fig. 5b) the relative error of NURBS elements is shown.

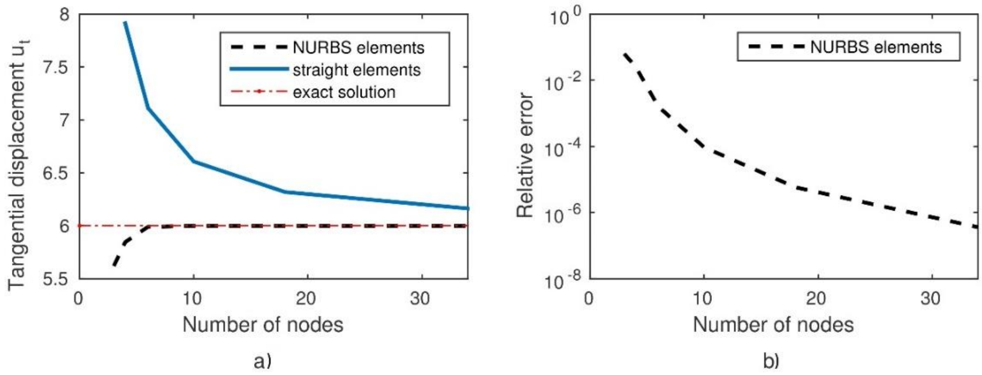

Fig. 5: Curved cantilever beam (sinusoidal moment loading): a) Comparison of tangential displacement of the tip obtained using traditional straight beam elements and NURBS elements. b) Dependence of the relative error of the NURBS elements solution on the number of nodes.

4. Conclusions

The formulation of NURBS beams has been presented. The element is formulated using NURBS approximation and therefore the exact geometry description is used for the analysis. The implementation enabled us to test the element and to compare its performance with traditional straight beam elements which use polynomial shape functions.

The necessity of linear parametrization to satisfy case of constant curvature is observed. Furthermore, while running the numerical tests, it has been proven that the element suffers from shear locking. The problem is solved by selective reduced integration. This treatment sufficiently removes shear locking and obtained results matches the expected ones.

Finally the curved structure has been analyzed. Superior convergence properties over the use of straight beam element is shown. Only small number of nodes is necessary to express exact solution and thus the computational cost is kept low.

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