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IMPLEMENTATION OF STABILIZED FORMULATION FOR LEVEL SET EQUATION

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Abstract: *This paper presents an implementation and verification of stabilized mass-conserving level set formulation. This formulation is suitable for further use in combination with XFEM for two phase flow simulations. The whole numerical scheme is based on finite element method and can be easily augmented into existing code for level set equation as a post-processing. Capability of the method is tested on a simple one dimensional numerical example.*

Keywords: Free-surface flow, Level set, Mass conservation, Fluids, Finite elements.

1. Introduction

This paper deals with implementation and verification of conservative level set method as an interface capturing technique for two-fluid flow problem. From a wider perspective, our interest lies in numerical simulations of fresh concrete casting, in particular with applications to Self-Compacting Concrete (SCC). Simulation of fresh concrete casting is naturally a free surface flow. We consider concrete as a single homogeneous fluid. Therefore, as it is standard in CFD, its motion is described within the Eulerian framework, in which free surface flow is modeled as a flow of two immiscible fluids in a fixed domain. One of the fluids represents the concrete, which is modeled as a non-Newtonian fluid using two-parametric Bingham model. The second fluid represents the air, which is modeled as a standard Newtonian fluid. In the context of Eulerian framework, description of the interface between the two fluids plays a key role in the whole problem and a proper interface-capturing method has to be chosen to deal with the interface. The problem can be solved using many well known techniques, for example level set method, volume of fluid method (VOF) and recently their combination, known as conservative level set and VOF method (CLSVOF). All of the aforementioned approaches have their own advantages and drawbacks. VOF based methods have superior conservation properties, while the interface reconstruction requires an additional effort (for example PLIC or least-squares techniques, see (Kees et al., 2011) for further reference). On the other hand, level set based methods can easily represent even very complicated interfaces, but suffer from being non-conservative, and an additional effort has to be made in order to enforce the conservation properties. However, for further use together with eXtended Finite Element Method (XFEM), the level set method is a natural choice. In the level set method, interface is represented implicitly as a zero level set of a suitable higher dimensional scalar function. In connection to XFEM, it is useful to choose that scalar function as a signed distance function, measuring the distance from the interface. Level set function with this property can be beneficially used in XFEM as a core of enriching approximation functions.

There are two main problems connected to level set method. One is that the level set function ϕ looses its signed distance property with evolution of the time. This represents a problem especially in combination with XFEM, where the signed distance property plays an important role. To reconstruct this property, proper reinitialization technique has to be employed. In this work, eikonal equation for the level set function is solved in order to recover the signed distance property. The second problem is that numerical schemes for evolution of the level set function generally do not conserve the mass. There are a few methods how to preserve the mass conservation, see Olsson & Kreiss (2005) for further reference. In this

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work, we employ an approach proposed in paper by see (Kees et al., 2011), where the evolution equation for the level set function is coupled with volume fraction equation through a correction to the level set function in order to preserve the mass conservation. This technique is described in the latter section.

2. Description of the Interface

In this section, description of the interface and its motion using level set method is given. In the level set method, the interface is described geometrically as a zero level set of an appropriate, higher dimensional scalar function ϕ (called level set function). Its motion due to the given velocity field is governed by following equation

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \phi = 0 \tag{1}$$

In general, ϕ can be chosen as an arbitrary differentiable function. However, especially in combination with XFEM, it is common to choose the level set function as a sign distance function, measuring the distance from the interface. In other words, ϕ has to satisfy following condition

$$\phi(\boldsymbol{x}) = \pm min_{x^* \in \Sigma} ||\boldsymbol{x} - \boldsymbol{x}^*|| \tag{2}$$

where Σ denotes the interface between both fluids. Numerical solution of equation (1) causes that level set function ϕ to gradually loose the signed distance property. This is a problem for future combination with XFEM, where the signed distance property plays an important role. The usual and widely applied way how to reinitialize the signed distance property is to solve iteratively the following equation

$$\frac{\partial \phi}{\partial \tau} = S(\phi_0)(1 - |\boldsymbol{\nabla} \phi|) \tag{3}$$

until it reaches the steady state. In the equation (3), $S(\phi_0)$ is regularized signum function, ϕ_0 represents level set function computed by (1) and is used as initial condition to (3). Parameter τ is a pseudo time.

The main problem of the numerical solution of (1) is that it is in general non-conservative and therefore it can produce unacceptable mass conservation errors. To solve this problem, we employ an approach proposed in a paper by (Kees et al., 2011). The level set equation is coupled with mass conservation equation through a correction ϕ' to the level set function ϕ. This approach is suitable for our purposes, since all the solutions are based on Galerkin formulation and could be easily solved by FEM. Therefore, the presented approach can be easily integrated into existing code for solving flow problems in OOFEM, see Patzák & Bittnar (2011). OOFEM is free finite element code with object oriented architecture for solving mechanical, transport and fluid mechanics problems, which is developed at Department of Mechanics, Faculty of Civil Engineering, CTU in Prague.

The mass conservation equation, which was mentioned above, can be expressed in terms of the Heaviside function $H(\phi)$. The total mass of fluid (denoted as *f*) can be computed as

$$M(t) = \int_{\Omega_f} \rho_f \, dx = \int_{\Omega} \rho_f H(\phi) \, dx \tag{4}$$

With this notion, the conservation of mass can be then expressed using the following continuity equation

$$\frac{\partial(\rho_f H(\phi))}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho_f H(\phi) \boldsymbol{v}\right) = 0 \tag{5}$$

In (4), $H(\phi)$ does not conserve the mass, since ϕ does not. If we denote the mass conserving approximation of volume fraction as $\widehat{H}$, then we want to compute a correction ϕ' to the level set function ϕ such that

$$\int_{\Omega} \left[H(\phi + \phi') - \widehat{H}\right] dx = 0 \tag{6}$$

To do so, authors (Kees et al., 2011) proposed to couple level set equation (1) and volume fraction equation (5) through the ϕ' into the following problem

$$\kappa \Delta \phi' = H(\phi + \phi') - \widehat{H} \tag{7a}$$

$$\boldsymbol{\nabla} \phi' \cdot \boldsymbol{n} = 0 \tag{7b}$$

The correction ϕ' is therefore computed as a post-processing step after the level set equation (1) is solved and it is added to ϕ in order to obtain corrected, mass conserving level set function.

3. Numerical Solution

In this section, numerical solution of the problem will be briefly described. In the latter, we assume that the velocity field, which arises in all presented equations, is given. Of course, in real simulations, the velocity field is obtained as a result of solving the Navier-Stokes equations. However, as it is not in the center of our attention, we will skip its precise description. Here, for the sake of brevity, the numerical procedure will be presented only in a schematical way. Few notes about the full discretization will be provided.

Given solution values at time t_n, we integrate the equations (1), (3), (5), (7) to time t_{n+1} gradually as follows. Assume that the velocity field $\boldsymbol{v}^{n+1}$ was already computed, we use it to advance the level set equation and volume fraction equation:

$$\frac{\phi_0^{n+1}-\phi^n}{\Delta t}+\boldsymbol{v}^{n+1}\cdot\nabla\phi_0^{n+1}=0 \tag{8}$$

$$\frac{\widehat{H}_0^{n+1}-\widehat{H}^n}{\Delta t}+\nabla\cdot\left(\widehat{H}_0^{n+1}\boldsymbol{v}^{n+1}\right)=0 \tag{9}$$

where ϕ_0^{n+1} and $\widehat{H}_0^{n+1}$ is the solution of the level set and volume fraction equation, respectively. Note that the computed level set function ϕ_0^{n+1} has lost the signed distance property and it does not conserve the mass in the sense that $H(\phi_0^{n+1})\neq\widehat{H}_0^{n+1}$. Next, the signed distance property of ϕ_0^{n+1} is recovered by solving the eikonal equation:

$$\|\phi_d^{n+1}\|=1 \tag{10a}$$

$$\phi_d^{n+1}=0\ \ on\ \Sigma_0=\{\boldsymbol{x}\in\Omega;\ \phi_0^{n+1}(\boldsymbol{x})=0\} \tag{10b}$$

The final step consists of solving nonlinear equation for mass-conservation correction ϕ' to ϕ_d^{n+1} as follows

$$H_\varepsilon(\phi_d^{n+1}+\phi')-\widehat{H}_0^{n+1}=\kappa\Delta\phi' \tag{11a}$$

$$\nabla\phi'\cdot\boldsymbol{n}=0\ on\ \partial\Omega \tag{11b}$$

Here, H_ε is regularized Heaviside function, ε is regularization parameter, which determines the width of smeared discontinuity and κ is another parameter, which penalizes deviation of ϕ' from global constant. After computation of ϕ', the final value of level set and volume fraction functions is given as

$$\phi^{n+1}=\phi_d^{n+1}+\phi' \tag{12}$$

$$\widehat{H}^{n+1}=H_\varepsilon(\phi_d^{n+1}+\phi') \tag{13}$$

Note that in equations (8)-(13), backward Euler method was used for the sake of simplicity. In our implementation, generalized mid-point rule is adopted. Spatial discretization of all presented equations is done in terms of finite element method. Note that because of the convective character of equations (8), (9) and (10) in the aforementioned form (3), additional stabilization technique has to be employed. In our case, SUPG and YZβ discontinuity capturing stabilizations were used. Precise description of spatial discretization will be skipped because of the abstract length limitation, interested reader can find the details in (Bazilevs et al., 2007).

4. Numerical Example

At this point, only simple 1D example, demonstrating capability of a prototype MATLAB implementation will be shown. More complex examples will be presented at the conference. To illustrate the problem, we consider a propagation of a square wave on a periodic domain for a volume fraction equation. This corresponds to propagation of a kink function on the same domain for the level set function, see Fig. 1. The problem can be described as follows

$$\Omega=[0,4] \tag{14a}$$

$$\widehat{H}(x,0) = \widehat{H}(\phi) \quad (14b)$$

$$\widehat{H}(0,t) = \widehat{H}(4,t) \quad (14c)$$

$$\phi(x,0) = \frac{1}{2} - |x-2| \quad (14d)$$

$$\phi(0,t) = \phi(4,t) \quad (14e)$$

The Fig. 1, illustrates the deformation of the level set profile for different numerical schemes for advecting the level set function. It can be seen, that stabilized Galerkin formulation provides totally incorrect solution. If the redistancing equation (10) is employed, the solution is much better, but from the Fig. 2 is clear, that for long time simulations, there is still a loss of mass. On the other hand, with the help of conservation correction equation (11), the total mass oscillates around the exact solution with error less than 0.1%.

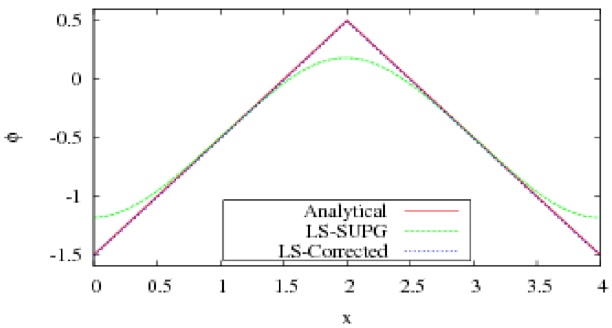

Fig. 1: Level set profiles after t = 10 s.

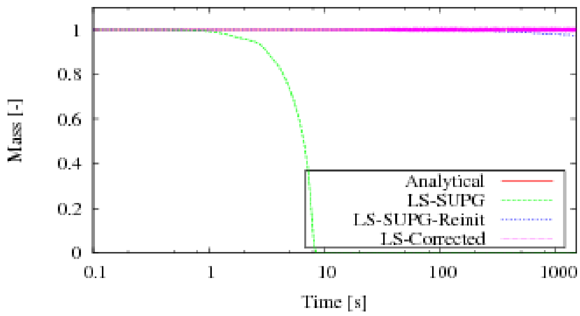

Fig. 2: Illustration of mass loss.

5. Conclusions

A prototype implementation and verification of stabilized mass-conserving level set formulation is presented. A simple numerical example is used for demonstration of the method's capability. It seems to be stable for long time simulations, which makes the method suitable for use in modeling of fresh concrete casting as a two fluid flow problem.

Future work lies in implementation of the method into OOFEM code.

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