20th International Conference
ENGINEERING MECHANICS 2014
Svratka, Czech Republic, 12 – 15 May 2014

A STABILITY CRITERION OF AN ORTHOTROPIC BI-MATERIAL NOTCH BASED ON THE STRAIN ENERGY DENSITY

M. Hrstka*, T. Profant, J. Klusák***, O. Ševeček****, M. Kotoul*****

Abstract: *A bi-material notch composed of two orthotropic materials is considered. The stress and displacement fields are expressed using the Lekhnitskii-Eshelby-Stroh formalism for plane elasticity and as the result of an application of the Ψ-integral method. Knowledge of the basic stress concentrator characteristics such as stress singularity exponent and corresponding regular and auxiliary eigenvectors allows the Ψ-integral to evaluate the generalized stress intensity factor of the studied stress concentrator. The achieved results are used for the next analysis such as assessment of the potential direction of the crack initiation and for notch stability criteria assessment as an extension of standard crack fracture mechanics.*

Keywords: **Fracture mechanics, Bi-material notch, Stability criteria, Singular stress concentrator, Stress intensity factor, Ψ-integral.**

1. Introduction

Joints of different materials, such as layered composite materials, fiber reinforced ceramics or construction with protective surface layer may occur in practical engineering structures. They enable to achieve the properties which could not be attained by means of homogeneous materials. In the case of composite materials, parts of the joints often exhibit orthotropic material properties. The stress field in closed vicinity of these material joints has a singular character and complicated form. In a comparison to a crack in homogeneous media, in the case of bi-material joints, the stress singularity exponent is different from $\mathbf{1/2}$ and generally can be complex. The stress is mostly characterized by more singular terms and at the same time each singular term covers combination of both normal and shear modes of loading (Broberg, 1999).

Despite a presence of stress concentrators like a notch, it precludes any application of the fracture mechanics approaches which were originally developed for a crack in isotropic homogeneous materials, the assessment of such singular stress concentrators becomes topical (Klusák & Knésl, 2007; Susmel & Taylor, 2008; Marsavina & Sadowski, 2008; Profant et al., 2010).

Most such discontinuities can be mathematically modeled as bi-material notches composed of two orthotropic materials given by angles $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$, as can be seen in Fig. 1a.

2. Materials and Methods

The necessary step for the crack initiation assessment is detailed knowledge of the stress distribution. Within plane elasticity of anisotropic media the Lekhnitskii-Eshelby-Stroh (LES) formalism based on complex potentials (Hwu, 2010) can be used. Complex potentials satisfying the equilibrium and the

* Ing. Miroslav Hrstka: Institute of Solid Mechanics, Mechatronics and Biomechanics, Brno University of Technology, Technická 2896/2, 616 69, Brno; CZ, y72539@stud.fme.vutbr.cz

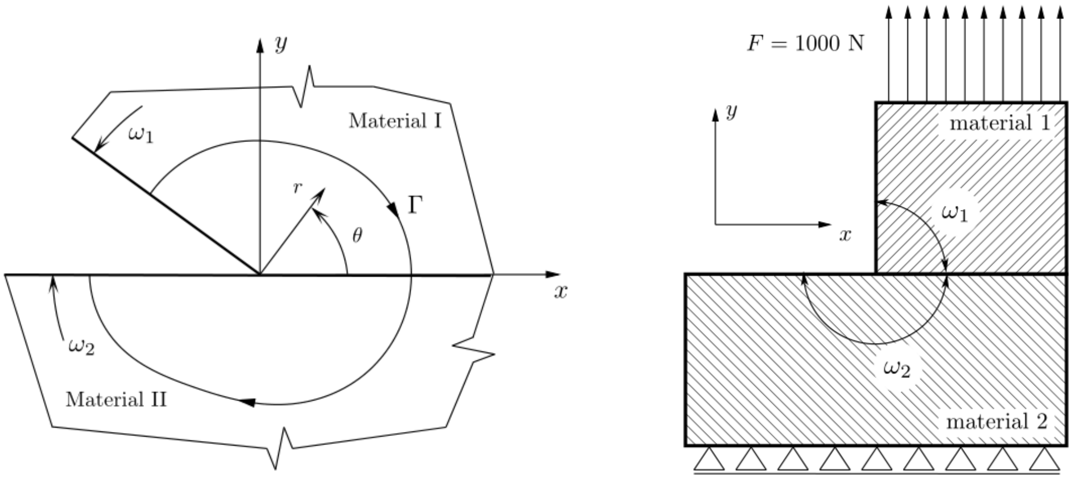
** Assoc. Prof. Ing. Tomáš Profant, PhD.: Institute of Solid Mechanics, Mechatronics and Biomechanics, Brno University of Technology, Technická 2896/2, 616 69, Brno; CZ, profant@fme.vutbr.cz

*** Assoc. Prof. Ing. Jan Klusák, PhD.: High Cycle Fatigue Group, Institute of Physics of Materials, Žižkova 22, 616 62, Brno; CZ, klusak@ipm.cz

**** Ing. Oldřich Ševeček, PhD.: Institute of Solid Mechanics, Mechatronics and Biomechanics, Brno University of Technology, Technická 2896/2, 616 69, Brno; CZ, sevecek@fme.vutbr.cz

***** Prof. RNDr. Michal Kotoul, DSc.: Institute of Solid Mechanics, Mechatronics and Biomechanics, Brno University of Technology, Technická 2896/2, 616 69, Brno; CZ, kotoul@fme.vutbr.cz

compatibility conditions as well as the linear stress-strain dependence and given boundary conditions are the basis for the determination of stress and deformation fields. In the case of general plane anisotropic elasticity all the components of the stress and deformation tensors have to be considered.



a) *b)*

Fig. 1: a) A bi-material notch given by angles $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$ *and composed of two orthotropic materials. An integration path* Γ *is surrounding the bi-material notch tip,* r *and* θ *are polar coordinates. b) A bi-material configuration with angles* ω_1=90°, ω_2=180° *and under external loading.*

2.1. Stress distribution

In terms of material with orthotropic properties, symmetry occurs in the stiffness and in the compliance matrices. Thus, the stress and strain tensor is significantly reduced. According to the LES theory for an orthotropic material, the relations for displacements and stresses can be written as follows

$$u_i = 2\Re\{A_{ij}f_j(z_j)\},\ \sigma_{2i} = 2\Re\{L_{ij}f_j^{'}(z_j)\},\ \sigma_{1i} = -2\Re\{L_{ij}\mu_j f_j^{'}(z_j)\}, \tag{1}$$

where $i, j = 1, 2$, $\Re$ denotes the real part of the complex expression and $z_j = x + \mu_j y$. Complex numbers μ_j are the eigenvalues of the material. For matrices A_{ij} and L_{ij} holds

$$\boldsymbol{A} = \begin{bmatrix} s_{11}\mu_1^2 + s_{12} & s_{11}\mu_2^2 + s_{12} \\ s_{12}\mu_1 + s_{22}/\mu_1 & s_{12}\mu_2 + s_{22}/\mu_2 \end{bmatrix},\ \boldsymbol{L} = \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix}, \tag{2}$$

where s_{ij} are the elastic compliances.

In the case of the studied notch, the potential $f_j(z_j)$ has the following form

$$\boldsymbol{f} = H\langle z_*^{\delta}\rangle \boldsymbol{v}\ , \tag{3}$$

re H is the generalized stress intensity factor, v_i is an eigenvector corresponding to the eigenvalue δ representing the exponent of the stress singularity at the notch tip. Eigenvector v_i and eigenvalue δ are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions. The expression $\langle z_*^{\delta}\rangle$ denotes a diagonal matrix $\langle z_*^{\delta}\rangle = \mathrm{diag}[z_1^{\delta},\ z_2^{\delta}]$. In most practical cases, there are two singular terms corresponding to two stress singularity exponents.

2.2. Determination of the generalized stress intensity factor

In order to determine the final stress distribution around the notch, it is important to find out the value of the generalized stress intensity factors (GSIF) from the analytical-numerical solution to a concrete situation with given geometry, materials and boundary conditions. The GSIFs can be determined using the so-called Ψ-integral (Kotoul et al., 2010; Hwu, 2010). This method is an implication of Betti's reciprocity theorem, which in the absence of body forces states that the following integral (4) is path-independent and equals

$$\Psi(\boldsymbol{u}, \widehat{\boldsymbol{u}}) = \int_{\Gamma} \{\sigma_{ij}(\boldsymbol{u})n_i\hat{u}_j - \sigma_{ij}(\widehat{\boldsymbol{u}})n_i u_j\}\,\mathrm{d}s,\ (i, j = 1, 2) \tag{4}$$

The contour Γ, as shown in Fig. 1a, surrounds the notch tip and the displacements $\boldsymbol{u}$ are considered as the regular and $\hat{\boldsymbol{u}}$ as the auxiliary solutions of the notch eigenvalue problem. For the corresponding exponents it holds $\hat{\delta} = -\delta$. If the contour Γ closely surrounds the notch tip, the Ψ-integral can be written as

$$\Psi(\boldsymbol{u}, \hat{\boldsymbol{u}}) = -\int_{-\omega_2}^{\omega_1} \{\varphi_{i,\theta}(\boldsymbol{u}) n_i \hat{u}_j - \varphi_{i,\theta}(\hat{\boldsymbol{u}}) n_i u_j\} \, \mathrm{d}\theta =$$

$$= H(c_1^I + c_2^I - c_3^I - c_4^I + c_1^{II} + c_2^{II} - c_3^{II} - c_4^{II}) \tag{5}$$

where the constants $c_1^I,\ldots,c_4^{II}$ are given by definite integrals independent of the coordinate r.

Because of their complicated form, they are not stated here. The superscript I or II corresponds to the material regions bounded by the angels ω_1 and ω_2. The comma in the subscript means the derivation with respect to θ. Because of the path independency of the integrals, the left hand side integral in (4) can be computed numerically along the path, which is any remote integration path with finite diameter. Since the exact solution $\boldsymbol{u}$ in this case is not known, a finite element solution can be used as its approximation.

2.3. Stability criterion

The stress field around a bi-material notch inherently covers combined normal and shear modes of loading. In the present paper where the two orthotropic materials are assumed as perfectly bonded, only crack propagation into materials I or II is supposed.

There are different methods to determine a stability criterion, for example as shown in (Erdogan & Sih, 1963; Profant et al. 2013). In present paper, a strain energy density factor (SEDF) is used, (Sih, 1977). Strain energy density is defined as

$$\Sigma(r, \theta) = r \frac{\mathrm{d}W}{\mathrm{d}V} = r \int_0^{\varepsilon_{pq}} \sigma_{ij} \mathrm{d}\varepsilon_{ij} \, , \tag{6}$$

where W is the strain energy, $\mathrm{d}V$ is diferential volume and ε_{ij} is strain. The integrand in (6) has to be total differential to provide the integral to be path-independent. Using the constitutive laws and substituting into (1) one can obtain

$$\Sigma(r, \theta) = \frac{1}{2} r^{2\delta_1 - 1} H_1^2 F_{H_1}(\theta) + r^{\delta_1 + \delta_2 - 1} H_1 H_2 F_{H_{1,2}}(\theta) + \frac{1}{2} r^{2\delta_2 - 1} H_2^2 F_{H_2}(\theta), \tag{7}$$

where the F_{H_1}, $F_{H_{1,3}}$, F_{H_2} are functions of the parameters δ and θ and can be found in (Profant et al., 2010). The strain energy density depends on the distance r from the notch tip. To make this dependence weaker, it is convenient to introduce a mean value of the SEDF over some distance d, which equals

$$\bar{\Sigma}(r, \theta) = \frac{1}{d} \int_0^d \Sigma(r, \theta) \, \mathrm{d}r \tag{8}$$

A potential crack can initiate in both materials and the crack direction is determined from the minimum of the mean value of the strain energy density. The resulting direction is then used for stability criterion estimation.

The critical value of the generalized stress intensity factor H_C is consequently determined from the strain energy density factor corresponding to the critical conditions, i.e. a crack initiates. A crack will not initiate at the tip of a bi-material notch if the value H is lower than its critical value H_C, i.e.

$$H(\sigma^{appl}) < H_C(K_{IC}), \tag{9}$$

where $\boldsymbol{\sigma^{appl}}$ is an applies load and $\boldsymbol{K_{IC}}$ the fracture toughness.

3. Results and Discussion

The procedure discussed above can be used to make a parametric study of the notch with geometry and external loadings given in Fig. 1b. The rectangular bi-material orthotropic notch is characterized by angles $\omega_1 = 90^\circ$ and $\omega_2 = 180^\circ$; this configuration often occurs in engineering constructions. The Young's moduli in Cartesian coordinate system according to the materials I, II are $(E_x)_I = 100$ MPa, $(E_y)_I = 50$ MPa,$(E_x)_{II} = 400$ MPa, $(E_y)_{II} = 50$ MPa. The eigenvalues corresponding to the stress singularity exponents are equal to $\delta_1 = 0.570$ and $\delta_2 = 0.939$ and GSIFs hold the values of $H_1 = 0.00348\,\mathrm{MPam}^{1-\delta_1}$ and $H_2 = 0.07039\,\mathrm{MPa\ m}^{1-\delta_2}$. The evaluation of SEDF $\bar{\Sigma}$ around the notch tip is depicted in the Fig. 2.

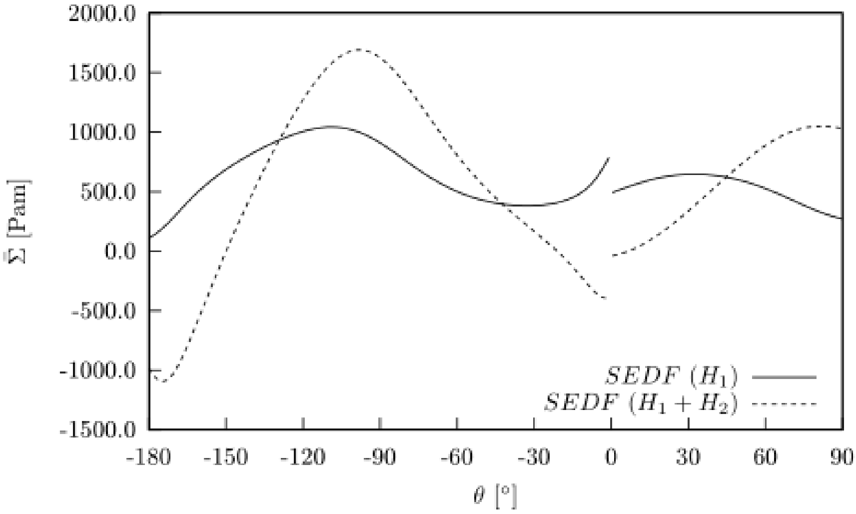


Fig. 2: Strain energy density factor as a function of the polar coordinate θ. The distance $d = 0.01$ mm. The minimal value determines the angle of crack initiation.

4. Conclusion

Composite structures involving interfaces, notches and free edges generally develop a singular elastic stress field near the intersection of lines of material and geometrical discontinuities. These localized regions of severe stress state are possible sites of failure initiation and growth. The aim of the present paper was to determine the stability criterion of the bi-material notch tip using a combination of analytical and numerical approach. As the stability criterion, the critical value of the generalized stress intensity factor was introduced. Using of this criterion is conditioned by the capable mathematical theory for the evaluation of those parameters, which characterize the stress singularity in notch tip vicinity and consequently allow to compute relevant concentrator characteristics. Because of the generalization of the stress singularity character at the notch due to its geometry, the Ψ-integral method and Lekhnitskii-Eshelby-Stroh were chosen.

Acknowledgement

The authors are grateful for financial support through the specific academic research grant of the Ministry of Education, Youth and Sports of the Czech Rep. No. FSI-S-14-2311 provided to Brno University of Technology, Faculty of Mechanical Engineering.

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