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FREE VIBRATION ANALYSIS OF TIMOSHENKO BEAM WITH DISCONTINUITIES USING DISTRIBUTIONS

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Abstract: *The general equations for the transverse vibration of Timoshenko beam have been used since they were derived by means of classical derivatives of the shear force, the bending moment, the rotation of a cross section and the deflection of the beam. However these derivatives are not defined at such points of a center-line between ends of the beam in which there is a concentrated support or a concentrated mass or a concentrated mass moment of inertia or an internal hinge connecting beam segments, which are discontinuities that can be met with in practice. We have applied distributional derivative for discontinuous shear force, discontinuous bending moment, and discontinuous rotation of a cross section of the beam in order to derive a generalized mathematical model for free transverse vibration as a system of partial differential equations. We have computed general solution to the generalized mathematical model for prismatic beam by means of symbolic programming approach via MAPLE. As a result of this approach, computing natural frequencies and modal shapes of the beam, we do not have to put together any continuity conditions at discontinuity points mentioned.*

Keywords: Timoshenko beam, Transverse vibration, Discontinuities, Dirac distribution.

1. Introduction

Classical analytical method of calculating natural frequencies of a beam with discontinuities is based on the following main steps (Timoshenko, 1937). Firstly we divide the beam into segments without discontinuities. Secondly we find continuous solution to a differential equation of motion for each segment separately. Thirdly we express boundary conditions for each segment, and continuity conditions among adjoining segments leading to a homogeneous system of linear algebraic equations. Finally we derive a frequency equation as a condition of nontrivial solution to the homogeneous system of linear equations.

Applying distributional derivative definition for discontinuous shear force, discontinuous bending moment, and discontinuous rotation of a cross section of a beam, we can derive a mathematical model for free transverse vibration of Timoshenko beam with discontinuities caused by concentrated supports or concentrated masses or concentrated mass moments of inertia situated between ends of the beam, or hinges connecting beam segments. This mathematical model can be solved like only one differential task without dividing the beam into segments where all the continuity conditions among adjoining segments are fulfilled automatically. Using this approach, we have only four integration constants irrespective of the number of the discontinuities.

2. Classical Equations of Motion for Free Transverse Vibration of Timoshenko Beam

According to Timoshenko's theory, we can express simultaneous differential equations of motion for free transverse vibration of the beam without discontinuities in the shear force, in the bending moment, in the rotation of the cross section or in the transverse displacement of the centerline of the beam (Rao, 2007) as

$$\rho\, \mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right) = k\, G\left(\frac{\partial}{\partial x}\left(\mathrm{A}(x)\left(\left(\frac{\partial}{\partial x}\mathrm{w}(x,t)\right) - \phi(x,t)\right)\right)\right), \quad (1)$$

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$$\rho\, \mathrm{J}(x) \left(\frac{\partial^2}{\partial t^2} \phi(x, t) \right) = k\, G\, \mathrm{A}(x) \left(\left(\frac{\partial}{\partial x} \mathrm{w}(x, t) \right) - \phi(x, t) \right) + E \left(\frac{\partial}{\partial x} \left(\mathrm{J}(x) \left(\frac{\partial}{\partial x} \phi(x, t) \right) \right) \right), \qquad (2)$$

where $\mathrm{w}(x, t)$ is the total transverse displacement of the beam centerline, $\phi(x, t)$ the rotation of the cross section assumed without warping, $\mathrm{A}(x)$ the cross-sectional area, $\mathrm{J}(x)$ the area moment of inertia, k the dynamic shear correction factor (Mindlin and Deresiewicz, 1953; Dong et al., 2010), E the modulus of elasticity (Young's modulus), G the shear modulus of elasticity, ρ the density.

3. Mathematical Model for Free Transverse Vibration of Timoshenko Beam with Discontinuities

In order to be able to express possible discontinuities in shear force, bending moment or in rotation of cross section along a centerline of a beam mathematically without cutting the beam into segments that would be without discontinuities in support, loading or without internal hinges, we use distributional derivative (Schwartz, 1966; Štěpánek, 2001; Kanwal, 2004), which consists of two main parts. Its first part is a classical derivative, while the second one is distributional as a sum of products of the Dirac singular distribution moved into the point of the discontinuity and a magnitude of the jump discontinuity of the quantity being differentiated.

When a beam supported at concentrated supports or carrying concentrated inertia masses between its ends is vibrating, jump discontinuities in shear force can occur at corresponding points of centerline of the beam. Expressing the first classical partial derivative of the shear force with respect to x from the force equation of motion for an element cut out of the beam, and adding distributional parts in the form of the product, we can derive Eq. (3), where $r_i(t)$ is a reaction force at i*th* concentrated support at a point $x = a_i$ ($0 < a_i < l$), l is the length of the beam, m_i is a concentrated inertia mass at a point $x = b_i$ ($0 < b_i < l$), Dirac(x-a_i) denotes the Dirac distribution moved into the point of the discontinuity, n_1 is a number of point supports, and n_2 is a number of concentrated inertia masses.

When a beam carrying concentrated masses with moments of inertia of J_i at points $x=c_i$ ($0 < c_i < l$) is vibrating, jump discontinuities in bending moment can occur at these points. Expressing the first classical partial derivative of the bending moment with respect to x from the moment equation of motion for an infinitesimal element of the beam, and adding products of a magnitude of the jumps and the Dirac distribution situated at the point of the discontinuity, we can acquire Eq. (4), the right hand side of which is the distributional derivative of the bending moment covering n_3 jump discontinuities.

If a beam containing hinges connecting segments of the beam at points $x=d_i$ ($0 < d_i < l$) is vibrating, jump discontinuities in rotation of the cross section of a magnitude $\psi_i(t)$ may be found at these points. Expressing the first classical partial derivative of the rotation of the cross section with the respect to x from the deformation relation of the beam centerline curvature, and adding corresponding distributional parts, we can obtain Eq. (5), where n_4 is a number of internal hinges.

$$\frac{\partial}{\partial x} \mathrm{Q}(x, t) = \rho\, \mathrm{A}(x) \left(\frac{\partial^2}{\partial t^2} \mathrm{w}(x, t) \right) + \left(\sum_{i=1}^{n_1} r_i(t)\, \mathrm{Dirac}(x - a_i) \right) + \left(\sum_{i=1}^{n_2} m_i \left(\frac{\partial^2}{\partial t^2} \mathrm{w}(x, t) \right) \Big|_{x=b_i} \mathrm{Dirac}(x - b_i) \right), \quad (3)$$

$$\frac{\partial}{\partial x} \mathrm{M}(x, t) = \mathrm{Q}(x, t) - \rho\, \mathrm{J}(x) \left(\frac{\partial^2}{\partial t^2} \phi(x, t) \right) - \left(\sum_{i=1}^{n_3} J_i \left(\frac{\partial^2}{\partial t^2} \phi(x, t) \right) \Big|_{x=c_i} \mathrm{Dirac}(x - c_i) \right), \qquad (4)$$

$$\frac{\partial}{\partial x} \phi(x, t) = -\frac{\mathrm{M}(x, t)}{E\, \mathrm{J}(x)} + \left(\sum_{i=1}^{n_4} \psi_i(t)\, \mathrm{Dirac}(x - d_i) \right), \qquad (5)$$

$$\frac{\partial}{\partial x} \mathrm{w}(x, t) = \phi(x, t) + \frac{\mathrm{Q}(x, t)}{k\, G\, \mathrm{A}(x)}\,. \qquad (6)$$

4. Free Vibration Solution

Supposing a harmonic time variation of solution to equations (3) to (6) as

$$\mathrm{Q}(x, t) = Q_s(x) \cos(\Omega\, t)\,, \qquad \mathrm{M}(x, t) = M_s(x) \cos(\Omega\, t)\,, \qquad \phi(x, t) = \phi_s(x) \cos(\Omega\, t)\,,$$

$$\mathrm{w}(x,t)=w_s(x)\cos(\Omega\,t)\,,\qquad r_i(t)=R_i\cos(\Omega\,t)\,,\qquad \psi_i(t)=\Psi_i\cos(\Omega\,t)\,,$$

where Ω is a circular frequency of vibration, and denoting amplitudes of vibration at points with concentrated transverse inertia forces and bending moments as

$$\mathrm{W}_i=\lim_{x\to b_i} w_s(x)\,,\qquad \Phi_i=\lim_{x\to c_i}\phi_s(x)\,, \tag{7}$$

we can derive a system of ordinary differential equations (8) to (11) for unknown general shapes of the deflection (w_s), the rotation of the cross section (ϕ_s), the bending moment (M_s), and the shear force (Q_s) for a uniform beam as

$$\frac{d}{dx}Q_s(x)=-\rho\,A\,w_s(x)\,\Omega^2+\left(\sum_{i=1}^{n_1}R_i\,\mathrm{Dirac}(x-a_i)\right)-\left(\sum_{i=1}^{n_2}m_i\,\mathrm{W}_i\,\Omega^2\,\mathrm{Dirac}(x-b_i)\right), \tag{8}$$

$$\frac{d}{dx}M_s(x)=Q_s(x)+\rho\,J\,\phi_s(x)\,\Omega^2+\left(\sum_{i=1}^{n_3}J_i\,\Phi_i\,\Omega^2\,\mathrm{Dirac}(x-c_i)\right), \tag{9}$$

$$\frac{d}{dx}\phi_s(x)=-\frac{M_s(x)}{E\,J}+\left(\sum_{i=1}^{n_4}\Psi_i\,\mathrm{Dirac}(x-d_i)\right), \tag{10}$$

$$\frac{d}{dx}w_s(x)=\phi_s(x)+\frac{Q_s(x)}{k\,G\,A}\,. \tag{11}$$

Characteristic functions of uniform Timoshenko beam with discontinuities

We have used the Laplace transform method so as to compute general solution to the system of Eqs. (8) to (11), i.e. characteristic functions of the beam, with integration constants in the form of initial parameters. Laplace transforms of unknown quantities are rational functions with a denominator which has a form of a quartic polynomial. Performing partial fraction decompositions of these rational functions, we must distinguish among three different cases:

i) $\Omega<\frac{\sqrt{J\,\rho\,A\,k\,G}}{J\,\rho}$,

ii) $\Omega>\frac{\sqrt{J\,\rho\,A\,k\,G}}{J\,\rho}$,

iii) $\Omega=\frac{\sqrt{J\,\rho\,A\,k\,G}}{J\,\rho}$.

In order to simplify expressions of the general solution, we introduce denotation:

$$\alpha=\frac{\sqrt{-2\,J\,E\,k\,G\,(\Omega\,J\,\rho\,k\,G+\Omega\,J\,E\,\rho-\sqrt{\Omega^2\,J^2\,\rho^2\,k^2\,G^2-2\,\Omega^2\,J^2\,\rho^2\,k\,G\,E+\Omega^2\,J^2\,E^2\,\rho^2+4\,J\,E\,k^2\,G^2\,A\,\rho}\,)\,\Omega}}{2\,E\,J\,G\,k},$$

$$\beta=\frac{\sqrt{2\,J\,E\,k\,G\,(\Omega\,J\,\rho\,k\,G+\Omega\,J\,E\,\rho+\sqrt{\Omega^2\,J^2\,\rho^2\,k^2\,G^2-2\,\Omega^2\,J^2\,\rho^2\,k\,G\,E+\Omega^2\,J^2\,E^2\,\rho^2+4\,J\,E\,k^2\,G^2\,A\,\rho}\,)\,\Omega}}{2\,E\,J\,G\,k}.$$

For example, when $\Omega<\frac{\sqrt{J\,\rho\,A\,k\,G}}{J\,\rho}$, the general shape of the rotation of the cross section may be expressed as follows:

$$\begin{aligned}\phi_s(x)=&\frac{(-\cosh(\alpha\,x)+\cos(\beta\,x))\,Q_s(0)}{E\,J\,(\alpha^2+\beta^2)}+\frac{((\rho\,\Omega^2-k\,G\,\beta^2)\sin(\beta\,x)\,\alpha-(\alpha^2\,G\,k+\rho\,\Omega^2)\sinh(\alpha\,x)\,\beta)\,M_s(0)}{(\alpha^2+\beta^2)\,\alpha\,\beta\,G\,k\,J\,E}\\&+\frac{(\cosh(\alpha\,x)\,(\alpha^2\,G\,k+\rho\,\Omega^2)+\cos(\beta\,x)\,(k\,G\,\beta^2-\rho\,\Omega^2))\,\phi_s(0)}{(\alpha^2+\beta^2)\,k\,G}\\&+\frac{A\,\rho\,\Omega^2\,(\beta\sinh(\alpha\,x)-\sin(\beta\,x)\,\alpha)\,w_s(0)}{(\alpha^2+\beta^2)\,E\,J\,\beta\,\alpha}\\&+\left(\sum_{i=1}^{n_1}\frac{\mathrm{Heaviside}(x-a_i)\,R_i\,(-\cosh(\alpha\,(x-a_i))+\cos(\beta\,(x-a_i)))}{J\,E\,(\alpha^2+\beta^2)}\right)\end{aligned}$$

$$+\left(\sum_{i=1}^{n_2} \frac{\text{Heaviside}(x-b_i)\, m_i\, W_i\, \Omega^2 (\cosh(\alpha\,(x-b_i)) - \cos(\beta\,(x-b_i)))}{E\,J\,(\alpha^2+\beta^2)}\right) + \left(\right.$$

$$\left.\sum_{i=1}^{n_3} \frac{\text{Heaviside}(x-c_i)\,(-(\alpha^2\,G\,k+\rho\,\Omega^2)\sinh(\alpha\,(x-c_i))\,\beta + (\rho\,\Omega^2 - k\,G\,\beta^2)\sin(\beta\,(x-c_i))\,\alpha)\,\Omega^2\,\Phi_i\,J_i}{(\alpha^2+\beta^2)\,\alpha\,\beta\,G\,k\,E\,J}\right)$$

$$+\left(\sum_{i=1}^{n_4} \frac{\text{Heaviside}(x-d_i)\,\Psi_i\,(\cos(\beta\,(x-d_i))\,(k\,G\,\beta^2-\rho\,\Omega^2) + \cosh(\alpha\,(x-d_i))\,(\alpha^2\,G\,k+\rho\,\Omega^2))}{(\alpha^2+\beta^2)\,k\,G}\right)$$

where Heaviside(*x*-*a*) is the denotation used in MAPLE for Heaviside's unit step function moved into the point $x = a$.

5. Conclusions

Contribution of this paper to modal analysis of Timoshenko beam is that the mathematical model for free transverse vibration, i.e. Eqs. (3) to (6), holds true also for the discontinuous shear force, the discontinuous bending moment and the discontinuous rotation of the cross section.

Discontinuities in shear force are supposed to be owing to idealized concentrated supports or inertia masses situated between ends of the beam. Likewise, discontinuities in bending moment are assumed to be due to idealized concentrated moments of inertia situated between ends of the beam. On the contrary, discontinuities in rotation of the cross section are caused by real hinges connecting beam segments. Jump discontinuities in unknown dependently variable quantities have been expressed in corresponding distributional derivatives (3)-(5), where the singular distribution Dirac(x), which is usually denoted as δ(x), is always moved into the point with the discontinuity mentioned, and multiplied by a magnitude of the discontinuity.

To be able to find natural frequencies and modal shapes of Timoshenko beam analytically with discontinuities mentioned, we have derived Eqs. (8) to (11) for shapes of the shear force, the bending moment, the rotation of the cross section and the deflection. Using the Laplace transform method with MAPLE software system, we have computed general solution to the system containing integration constants in the form of initial parameters. Computing limits (7), we can express the unknown amplitudes of the deflection, W_i, and rotation of the cross section, Φ_i, as functions of initial parameters. In order to determine the unknown initial parameters, we must establish four boundary conditions. So as to determine the unknown reactions at concentrated supports between ends of the beam, and amplitudes of discontinuities in the rotation of the cross section at hinges connecting beam segments, we must establish corresponding deformation conditions at these points. These deformation and boundary conditions create all together a homogeneous system of linear equations. The condition of the nontrivial solution to this system is the frequency equation of the beam with discontinuities assumed.

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