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NUMERICAL MODELING OF GROUNDWATER FLOW IN RANDOM MATERIALS

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Summary: *The prediction of groundwater flow is strongly influenced by the soil permeability generally varying within the space. Determination of the spatial distribution of the permeability is, however, unfeasible and thus the relevant uncertainties should be taken into account. One possibility is to describe the soil permeability by a random field. The present contribution is devoted to propagation of these uncertainties in permeability into probabilistic description of groundwater flow.*

1. Introduction

This paper is focused on the modeling of uncertainties in material properties and investigates the influence of such uncertainties on groundwater flow, described by a steady-state diffusion equation. As a simple example, consider the following (deterministic) elliptic partial differential equation (PDE) for the hydraulic head $u(x)$:

$$-\nabla \cdot (\kappa(x)\nabla u(x)) = f(x), \qquad x \in D, \tag{1}$$

$$u(x) = g(x), \qquad x \in \partial D, \tag{2}$$

where $\kappa(x)$ is the soil permeability (hydraulic conductivity), $f(x)$ is a given source or sink inside the region D ($D \subset \mathbb{R}^2$) and $g(x)$ are prescribed flows and hydraulic heads on the boundary ∂D.

Consider now a system involving material variability. If the input parameter is defined as a random field, the system would be governed by a set of stochastic partial differential equations (SPDE) and the corresponding responses would also be random vectors of nodal displacements, see (Keese and Matthies, 2005; Kučerová and Sýkora, 2013). Let us formulate this for the soil permeability $\kappa(x)$. A random model is obtained by defining $\kappa(x)$ for each $x \in D$ as a random variable $\kappa(x) : \Omega \to \mathbb{R}$ on a suitable probability space $(\Omega, \mathscr{S}, \mathbb{P})$. As a consequence, $\kappa : D \times \Omega \to \mathbb{R}$ is a random field, where any elementary event $\omega \in \Omega$ gives a realization $\kappa(\cdot, \omega) : D \to \mathbb{R}$ of the soil permeability. Alternatively, $\kappa(x, \omega)$ can be seen as a collection of real-valued random variables indexed by $x \in D$, see (Keese and Matthies, 2005; Keese, 2004; Eiermann et al., 2007). Introduction of random system parameters into Eqs. (1) and (2)

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we obtain the stochastic partial differential equation:

$$-\nabla \cdot (\kappa(x,\omega)\nabla u(x,\omega)) = f(x,\omega), \qquad x \in D, \tag{3}$$

$$u(x,\omega) = g(x,\omega), \qquad x \in \partial D. \tag{4}$$

In order to solve this stochastic partial differential equation and obtain the approximate responses of the system, Monte Carlo (MC) method is usually used. The effort of performing Monte Carlo simulations is high, and hence alternative techniques have been developed, such as the spectral stochastic finite element method (SSFEM). The interested reader is referred to (Keese and Matthies, 2005; Kučerová et al., 2012; Chen and Soares, 2008) for further information.

2. Discretization of random fields

Assuming the random field $\kappa(x,\omega)$ to be Gaussian, it is defined by its mean

$$\mu_\kappa(x) = \mathbb{E}[\kappa(x,\omega)] = \int_\Omega \kappa(x,\omega)\,\mathbb{P}(\mathrm{d}\omega) \tag{5}$$

and its covariance

$$\begin{aligned} C_\kappa(x,x') &= \mathbb{E}[(\kappa(x,\omega)-\mu_\kappa(x))(\kappa(x',\omega)-\mu_\kappa(x'))] \\ &= \int_\Omega (\kappa(x,\omega)-\mu_\kappa(x))(\kappa(x',\omega)-\mu_\kappa(x'))\,\mathbb{P}(\mathrm{d}\omega)\,. \end{aligned} \tag{6}$$

In a computational setting, the random field and the numerical model must be discretized. The most common approach for achieving this is the Karhunen-Loéve expansion (KLE), see (Eiermann et al., 2007). KLE is an extremely useful tool for the concise representation of the stochastic processes. Based on the spectral decomposition of covariance function $C_\kappa(x,x')$ and the orthogonality of eigenfunctions ψ_i, the random field $\kappa(x,\omega)$ can be written as

$$q(x,\omega) = \mu_\kappa(x) + \sum_{i=0}^{\infty} \sqrt{\varsigma_i}\xi_i(\omega)\psi_i(x), \tag{7}$$

where $\boldsymbol{\xi}(\omega) = (\ldots,\xi_i(\omega),\ldots)^{\mathrm{T}}$ is a set of uncorrelated random variables of zero mean and unit variance. The spatial KLE functions $\psi_i(\boldsymbol{x})$ are the eigenfunctions of the Fredholm integral equation with the covariance function as the integral kernel:

$$\int_D C_\kappa(x,x')\psi_i(x)\mathrm{d}x' = \varsigma_i\psi_i(x)\,, \tag{8}$$

where ς_i are positive eigenvalues ordered in a descending order.

Since the covariance is symmetric and positive definite, it can be expanded in the series

$$C_\kappa(x,x') = \sum_{i=1}^{\infty} \varsigma_i\psi_i(x)\psi_i(x')\,. \tag{9}$$

However, computing the eigenfunctions analytically is usually not feasible. Therefore, one discretizes the covariance spatially according to chosen grid points (usually corresponding to

a finite element mesh). The resulting covariance matrix C_κ is again symmetric and positive definite and Eq. (8) becomes symmetric matrix eigenvalue problem, see (Keese and Matthies, 2005), where the eigenfunctions $\psi_i(x)$ are replaced by eigenvectors $\boldsymbol{\psi}_i$. The eigenvalue problem is usually solved by a Krylov subspace method with a sparse matrix approximation. For large eigenvalue problems, (Khoromskij and Litvinenko,, 2008) proposes the efficient low-rank and data sparse hierarchical matrix techniques.

For practical implementation, the series (7) and (9) are truncated after M terms, yielding the approximations

$$\hat{\boldsymbol{\kappa}}(\omega) \approx \mu_\kappa + \sum_{i=1}^{M} \sqrt{\varsigma_i}\xi_i(\omega)\boldsymbol{\psi}_i\,, \tag{10}$$

$$\hat{\mathrm{C}}_\kappa \approx \sum_{i=1}^{M} \varsigma_i \boldsymbol{\psi}_i^{\mathrm{T}} \cdot \boldsymbol{\psi}_i\,. \tag{11}$$

Such spatial semi-discretization is optimal in the sense that the mean square error resulting from a truncation after the M-th term is minimized.

3. Numerical study

This section supports through numerical study the proposed methodology. In doing so we consider geometry together with the initial and loading conditions displayed in Fig. 1. 2-D domain was discretized by an FE mesh into 1195 nodes and 2228 triangular elements.

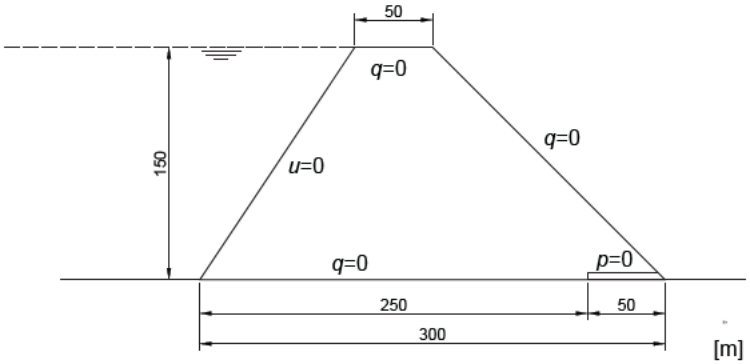


Figure 1: 2-D domain with boundary conditions

It is clear from the preceding text that the implementation of the Karhunen-Loève expansion requires knowing the covariance function of the process being expanded. We assume the normalized exponential covariance kernel described by following formula:

$$C_\kappa = \exp\left(-\left|\frac{x-x'}{l_x}\right| - \left|\frac{y-y'}{l_y}\right|\right), \tag{12}$$

where $l_x = 5$ [m] and $l_y = 3$ [m] are covariance lengths. In practise, the correlation lengths can be determined by the image analysis of a given material (Lombardo et al., 2010). Several interesting results have been derived within the scope of the calculation of KLE. Associated eigenvectors are collected in Figs. 2(a)-(d) and Figs. 3(a)-(b) presents a comparison of an arbitrary

realization of hydraulic field $\kappa(x)$ computed using all 1195 eigenmodes and its approximation $\hat{\kappa}(x)$ computed using only first 100 eigenmodes.

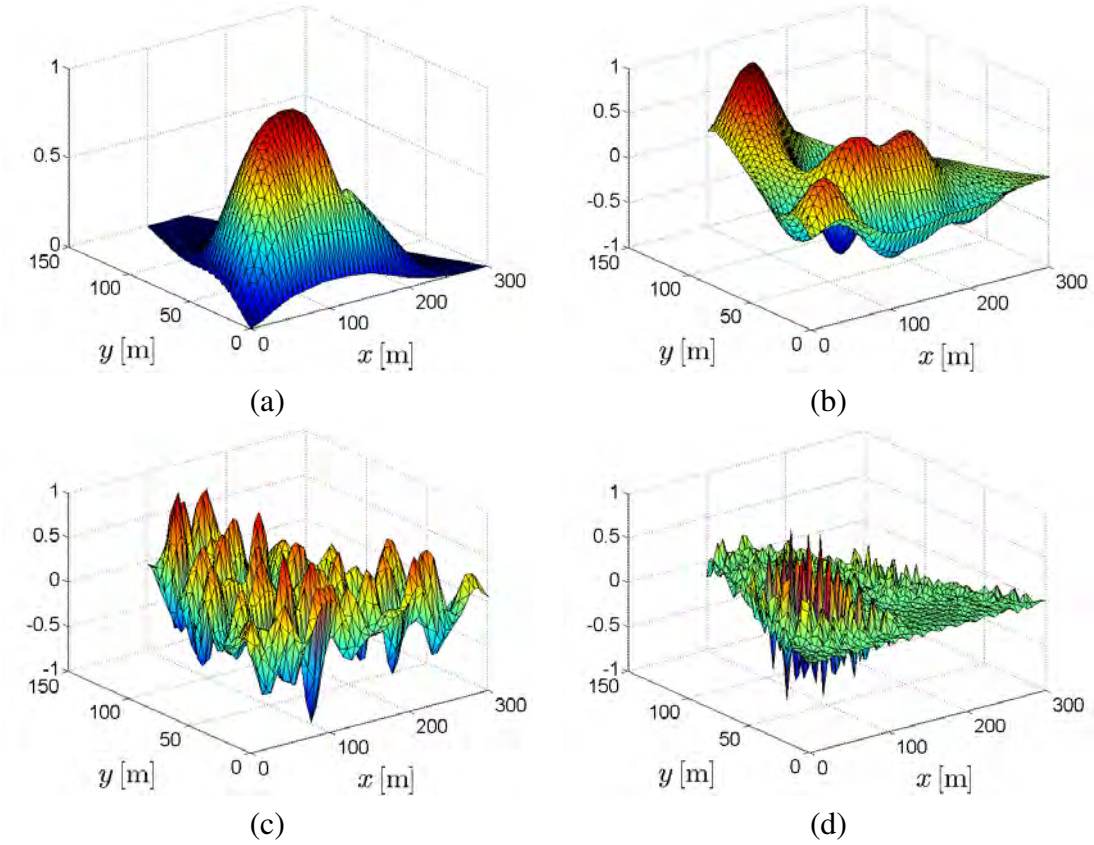


Figure 2: Examples of eigenvectors ψ_i, (a) $i = 1$, (b) $i = 10$, (c) $i = 100$, (d) $i = 1195$

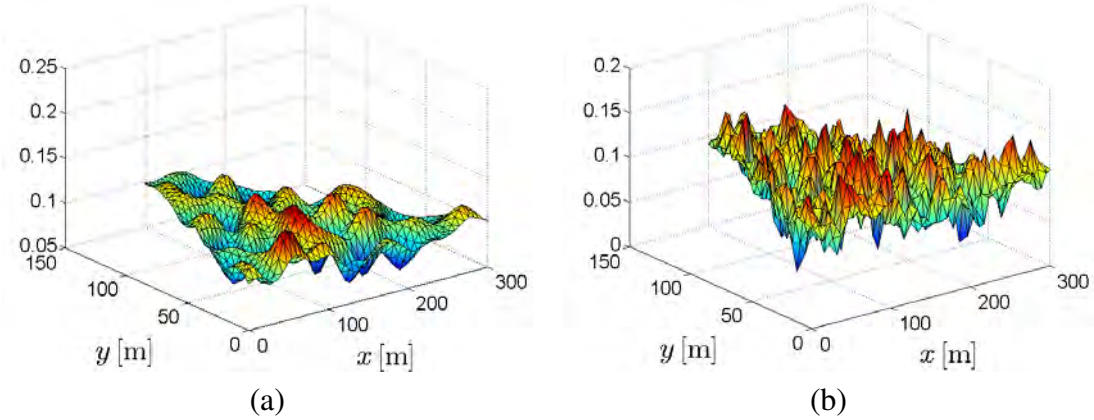
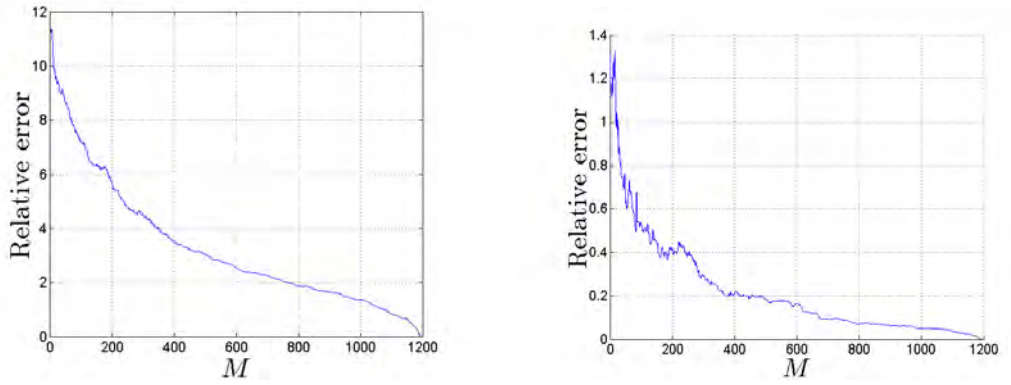


Figure 3: The hydraulic conductivity field computed using, (a) only first 100 eigenmodes, (b) all 1195 eigenmodes

In order to choose an appropriate number of eigenmodes, a relative pointwise error of input fields averaged over all finite elements and over independent random realizations was computed. A similar error can be also computed in terms of response fields. These errors as a function of

the number of eigenmodes M involved in the description of input fields are depicted in Fig. 4. It can be seen again that the error in description of input fields is decreasing slowly, while the error in the response fields descends much faster due to the smoothing effect of the numerical model.



(a) (b)

Figure 4: Relative mean point-wise error [%], (a) of the input hydraulic conductivity field and (b) of the overall responses induced by KLE approximation based on M eigenmodes

4. Conclusion

This paper presents the numerical study of groundwater flow in random media under steady-state conditions. For a sake of simplicity, the random fields are assumed to be Gaussian.

All numerical simulations were performed using Monte Carlo technique, which is very computationally exhaustive procedure. The spectral stochastic finite element method is promising alternative. It is under current investigation and will be presented elsewhere.

5. Acknowledgment

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