19th International Conference
ENGINEERING MECHANICS 2013
Svratka, Czech Republic, May 13 – 16, 2013

BRANCH AND BOUND METHOD FOR OPTIMIZATION OF CABLE-TRUSS STRUCTURES

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Summary: *This contribution introduces a new formulation of a classical Branch and Bound method to find global optima of truss sizing optimization problems enhanced with cables. Our paper shows that the problem is solvable as a binary version of the mixed integer programming problem. The text presents the derivation and the implementation details.*

Keywords: ***Size Optimization, Cable-Truss Structures, Benchmarks, Global Optima, Mixed-Integer Linear Problem, Big-M Problem, Cables***

1. Introduction

Many optimization algorithms based on different principles have been developed in recent years. To compare them, different testing functions and structures, called benchmarks[2], are frequently used. Nevertheless, most of the authors commonly uses the same version to compare their optimization methods. For comparing them, it is good to know properties of these benchmarks. One of them are global optima.

It is very useful to know global optima to determine how the tested algorithm acts like. If the optimization method gets closer to the global optimum in a short time and do not terminated in the local optimum is consider as a sign of quality. In our previous contribution (Pospíšilová, 2013), the derivation of the search for global optima for classical sizing optimization benchmarks has been presented. These benchmarks share same qualities such as the objective is a linear combination of variables and thus it is easy to solve, constraints are non-linear and the searched space is discrete. This contribution then presents a next steps that are needed to solve the global optima of the Cable-Truss Structures. These are characterized by sets (groups) of cross-sectional areas that differ in length, i.e. usually one group for bars and one for cables, respectively. Moreover, the different behavior of cables in compression must be taken into account.

For getting global optimum in a real time on types of benchmarks defined above, it is possible to use few methodologies. An Exhaustive Search (also called Enumeration or Brutal Force, respectively) is one of them. It is possible to use it just for very small examples thus it is

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[2] For structural optimization, there are about ten truss-like structures that are used in slightly modified versions. These modifications are not comparable to each others which is the common mistake of some authors. For more details of these modifications of some very often used benchmarks see Pospíšilová (2011).

necessary to evaluate objective and constraints for every combination of variables. The problem grows exponentially with increasing variables or conditions. Other approaches are Branch and Bound Methods (Arora, 2002), Branch and Cut Methods (Mitchell, 2002) or Methods Based on Branch and Bound Principles combined with the Enumeration (Pospíšilová, 2012). The last mentioned procedure is based on the idea, that the optimization process systematically goes through the searched space, that is delimited by two known bounds in advance. The lower bound is invariable and it is obtained by continuous optimization (e.g. Active Set Method). The upper bound is obtained by any heuristic or metaheuristic method. The best-so-far solution is used to reduce the searched space from above by updating the upper bound. The algorithm ends after searching the whole space in between. The advantage of this method is that the global optimum is found for every typecast benchmark. The disadvantage is that the methodology is time consuming even if it is parallelized. In this paper, our attention is dedicated to the classical Branch and Bound Method, that could be less time consuming than the mentioned approach.

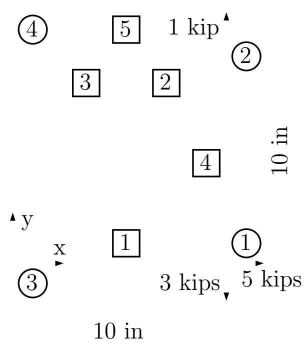

Figure 1: 5-bar truss structure

2. Size optimization

Sizing optimization (Bendsøe, 1995) is one type of structural optimization that deals with truss-like structures. These structures are defined by topology, material, loading, supports, and a set of cross-sections or, alternatively, minimum and maximum cross-sectional areas of the individual rods. The objective function is the weight of a structure and constraints are maximal stresses and maximal displacements, respectively. The goal is to find cross-sections for the given structure that satisfy prescribed constraints and have the minimal weight. The selection of cross-sections from the given database then defines a discrete optimization problem, whereas variables chosen from given limits leads to a continuous case. The continuous optimization problem can be efficiently solved by mathematical programming methods like gradient-based methods. These methods do not guarantee finding the global optimum for non-convex problems however, they often terminated in a critical point. When using discrete variables, no such option is available. Thus our attention is given to the discrete case.

3. Relaxation of the Problem

The truss structure is defined in sense of Fig. 1. The topology of the structure is unchanging and the goal is to find minimal weight with the best combination of cross-section areas. Prescribed constraints are minimal and maximal possible values of displacements as well as stresses.

An optimization problem is possible to be defined as (Rasmussen, 2008)

$$\min_{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\mathbf{u}\in\mathbb{R}^{nd}} \quad \rho\sum_{j=1}^{nr}\ell_j\sum_{i=1}^{na}a_i x_{i,j} \quad \text{(weight)} \tag{1}$$

$$\text{subject to} \quad \mathbf{K}(\mathbf{x})\mathbf{u}=\mathbf{f} \quad \text{(force equilibrium)}, \tag{2}$$

$$\sum_{i=1}^{na}x_{i,j}=1 \quad \forall j \quad \text{(one area per bar)} \tag{3}$$

$$\mathbf{u}^{\min}\leq\mathbf{u}\leq\mathbf{u}^{\max} \quad \text{(displacement constraint)} \tag{4}$$

$$x_{i,j}\in 0,1 \quad \forall(i,j). \tag{5}$$

Notation (1) describes minimization of the structure weight where ρ is the density of the material, ℓ_j is the length of the j^{th} rod, a_i is the i^{th} cross-sectional area and $x_{i,j}$ is the binary design variable that is arranged in a column vector. The variable i assumes values between 1 and na where na is a number of areas in the given set. The variable j is defined with values 1 to nr where nr is a number of rods of the structure. $x_{i,j}$ is 1 whether area a_i is assigned to j^{th} rod. In other case, $x_{i,j}$ is 0. Equation (2) is force equilibrium defined with the stiffness matrix $\mathbf{K}(\mathbf{x})$ with size $nd\times nd$, the displacement vector $\mathbf{u}$ with length nd and the loading vector $\mathbf{f}$ with length nd. nd is a number of degrees of freedom. Equation (3) ensures that only one area is assigned to each rod. Inequality (4) is the displacement constraint condition ensuring that values of variables stays in a feasible region. Formula (5) guarantees that $x_{i,j}$ can gained 0 and 1 values.

An optimization problem described above is called mixed-integer non-linear problem. Because of the force equilibrium (2) which is non-convex, there is no guarantee that optimum is possible to find. This nonlinear problem can be transformed into pure linear one (Rasmussen, 2008):

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \quad \rho\sum_{j=1}^{nr}\ell_j\sum_{i=1}^{na}a_i x_{i,j} \quad \text{(weight)} \tag{6}$$

$$\text{s.t.} \quad \mathbf{B}\mathbf{s}=\mathbf{f} \quad \text{(force equilibrium)}, \tag{7}$$

$$x_{i,j}a_i\sigma^{\min}\leq s_{i,j}\leq x_{i,j}a_i\sigma^{\max}\ \forall(i,j)\ \text{(stress constraints)}, \tag{8}$$

$$\frac{E_j a_i}{\ell_j}\mathbf{b}_j^{\mathrm{T}}\mathbf{u}-s_{i,j}\geq(1-x_{i,j})c_{i,j}^{\min}\ \forall(i,j)\ \text{(compatibility)}, \tag{9}$$

$$\frac{E_j a_i}{\ell_j}\mathbf{b}_j^{\mathrm{T}}\mathbf{u}-s_{i,j}\leq(1-x_{i,j})c_{i,j}^{\max}\ \forall(i,j)\ \text{(compatibility)}, \tag{10}$$

$$\sum_{i=1}^{na}x_{i,j}=1 \quad \forall j \quad \text{(one area per bar)}, \tag{11}$$

$$\mathbf{u}^{\min}\leq\mathbf{u}\leq\mathbf{u}^{\max} \quad \text{(displacement constraints)}, \tag{12}$$

$$x_{i,j}\in\{0,1\} \quad \forall(i,j). \tag{13}$$

Geometric matrix $\mathbf{B}$ containts vectors $\mathbf{b}_j$ which are the direction cosine vectors, s are internal normal forces, $\mathbf{f}$ is loading vector. $\sigma^{\min}$ and $\sigma^{\max}$ are minimal and maximal permitted values of stresses, E_j is Young modulus, $s_{i,j}$ are all possible internal normal forces for every combination of cross-sectional area on all rods, $\mathbf{u}$ is the displacement vector, $c_{ij}^{\min}$ and $c_{i,j}^{\max}$ are minimal and maximal possible internal forces, respectively.

For implementation purposes, the notation is possible to be rewritten to

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \quad \rho\Big((\boldsymbol{\ell}\otimes\mathbf{1}_{[na]})^T \odot (\mathbf{1}_{[nr]}\otimes\mathbf{a})^T\Big)\,\mathbf{x} \tag{14}$$

$$\text{s.t.} \quad \mathbf{B}\mathbf{s} = \mathbf{f}, \tag{15}$$

$$\sigma^{\min}\operatorname{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{x} \le \mathbf{s} \le \sigma^{\max}\operatorname{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{x}, \tag{16}$$

$$\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u} - \mathbf{s} \ge \big(\mathbf{1}_{[na\cdot nr]} - \mathbf{x}\big)\odot\mathbf{C}^{\min}, \tag{17}$$

$$\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u} - \mathbf{s} \le \big(\mathbf{1}_{[na\cdot nr]} - \mathbf{x}\big)\odot\mathbf{C}^{\max}, \tag{18}$$

$$\mathbf{I}_{[nr]}\otimes\mathbf{1}_{[na]}\cdot\mathbf{x} = \mathbf{1}_{[nr]}, \tag{19}$$

$$\mathbf{x}\in\{0,1\}. \tag{20}$$

Vector $\boldsymbol{\ell}$ contains all lengths of rods, all cross-sections are arranged in a vector $\mathbf{a}$, vector $\mathbf{x}$ contains all variables $x_{i,j}$, ξ is a specific stiffness, $\mathbf{s}$ is a vector of internal normal forces, $\mathbf{C}^{\min}$ and $\mathbf{C}^{\min}$ are matrices of maximal and minimal possible internal forces respectively, $\mathbf{I}$ is an identity matrix, $\mathbf{1}$ is a vector or a matrix filled with 1. The specific stiffness ratio ξ is a product of the Young's modulus and a vector of multiplicative inverse elements to length vector $\boldsymbol{\ell}$, i.e. $\xi = E\cdot\bar{\boldsymbol{\ell}}$, where $\bar{\boldsymbol{\ell}}\cdot\frac{1}{\boldsymbol{\ell}} = 1$. For definition of the $\odot$ and $\otimes$ operators, inspect Appendix A. For more details on the derivation, inspect again (Pospíšilová, 2013).

4. Trusses with groups

Several benchmarks does not use all cross-sectional areas as variables. Instead of it, rods are divided into groups and these groups are optimization variables. All binary variables $\mathbf{x}_{[ng\cdot na]}$ has to be increased with the extended localization matrix $\mathbf{G}_{[nr\cdot na\times ng\cdot na]}$ as $\mathbf{G}\cdot\mathbf{x}$ that gives a vector with the size $[nr\cdot na]$. ng is a number of groups. The extended localization matrix $\mathbf{G}$ is Kronecker product of localization matrix $\mathbf{Q}_{[nr\times ng]}$ and identity matrix $\mathbf{I}_{[na]}$ ($\mathbf{G} = \mathbf{Q}\otimes\mathbf{I}_{[na]}$). The localization matrix $\mathbf{Q}$ is a full matrix of ones and zeros. Element $q_{p,r}$ is equal to one whether the corresponding bar on position p is assigned into the corresponding group on possition r, i.e. $q_{3,2} = 1$ as the second group is assigned to the third bar, the rest of the third row in the $\mathbf{Q}$ matrix is filled with zeros due to the condition that only one area has to be assigned per bar.

The resulting system then reads

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \quad \rho\Big((\boldsymbol{\ell}\otimes\mathbf{1}_{[na]})^T\odot(\mathbf{1}_{[nr]}\otimes\mathbf{a})^T\Big)\,\mathbf{G}\mathbf{x} \tag{21}$$

$$\text{s.t.} \quad \mathbf{B}\mathbf{s}=\mathbf{f}, \tag{22}$$

$$\sigma^{\min}\operatorname{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{G}\mathbf{x}\le\mathbf{s}\le\sigma^{\max}\operatorname{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{G}\mathbf{x}, \tag{23}$$

$$\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\ge\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{G}\mathbf{x}\big)\odot\mathbf{C}^{\min}, \tag{24}$$

$$\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\le\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{G}\mathbf{x}\big)\odot\mathbf{C}^{\max}, \tag{25}$$

$$\mathbf{I}_{[nr]}\otimes\mathbf{1}_{[na]}\cdot\mathbf{G}\mathbf{x}=\mathbf{1}_{[nr]}, \tag{26}$$

$$\mathbf{x}\in\{0,1\}. \tag{27}$$

Nevertheless, there can be different sets of cross-sectional areas set up for different rods. Let us assume five bar truss example in Fig.1 with two different cross-section sets. Cross-sections from the first set $\hat{\mathbf{a}}$ can be assigned to the vertical and chords (namely bars 1, 4, and 5) and cross-sections from the second set $\hat{\hat{\mathbf{a}}}$ to diagonals (namely bars 2 and 3). Sets can be also created with a different number of cross-sections, e.g. $\hat{\mathbf{a}}=(a_1,a_2)^{\mathrm{T}}$ and $\hat{\hat{\mathbf{a}}}=(a_3,a_4,a_5)^{\mathrm{T}}$. However, that kinds of sets with various sizes are uncomfortable to maintain thus the smaller sets are increased to the size of the biggest set with sections that are as large as the solution that is not going to be used for branching the problem. This virtual section is denoted by $\bar{a}$, thus both sets look like $\hat{\mathbf{a}}=(a_1,a_2,\bar{a}_3)^{\mathrm{T}}$ and $\hat{\hat{\mathbf{a}}}=(a_4,a_5,a_6)^{\mathrm{T}}$. To assign proper sections to proper rods, the matrix $\mathbf{V}$ is introduced. A number of rows of $\mathbf{V}$ is similar to number of rods nr and a number of columns is the same as a number of sets ns. Every column corresponds to a different set. Element $v_{j,q}$ is equal to one whether the cross-section on the j^{th} rod is chosen from the q^{th} set otherwise it is equal to zero. For the sake of clarity, note that now na is as big as the number of sections in the biggest set.

To evaluate the objective function that has to be minimized, the equation (14) is redefined as

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \rho\Big((\boldsymbol{\ell}\otimes\mathbf{1}_{[na]})^T\odot(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{a})^T\Big)\mathbf{x} \tag{28}$$

where the vector $\mathbf{a}$ contains both sets of cross-sections. For the five bar truss problem with two sets, $\mathbf{a}$ is composed as $(\hat{\mathbf{a}},\hat{\hat{\mathbf{a}}})$. Stress constraints from equation (16) and compatibility equations (17) and (18) are redefined in the same meaning. The resulting system then leads to

$$\sigma^{\min}\operatorname{diag}\big(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{a}\big)\,\mathbf{x}\le\mathbf{s}\le\sigma^{\max}\operatorname{diag}\big(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{a}\big)\,\mathbf{x}, \tag{29}$$

$$\Big(\big(\xi\otimes\mathbf{1}_{[na]}\odot(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{a})\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\ge\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{x}\big)\odot\mathbf{C}^{\min}, \tag{30}$$

$$\Big(\big(\xi\otimes\mathbf{1}_{[na]}\odot(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{a})\otimes\mathbf{1}_{[nd]}^T\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\le\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{x}\big)\odot\mathbf{C}^{\max}. \tag{31}$$

Constraint equalities (15) and (19) and definition of binary decision variable (20) remain unchanged. Additional constraints are defined as

$$\operatorname{diag}\big(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{z}\big)\cdot\mathbf{x} \;=\; \mathbf{0}_{[na\times nr]}, \tag{32}$$

$$\operatorname{diag}\big(\mathbf{V}\otimes\mathbf{I}_{[na]}\cdot\mathbf{z}\big)\cdot\mathbf{s} \;=\; \mathbf{0}_{[na\times nr]}, \tag{33}$$

where vector $\mathbf{z}$ holds information about the added virtual cross-section. It is composed by two vectors $\hat{\mathbf{z}}$ and $\hat{\hat{\mathbf{z}}}$ ($\mathbf{z} = (\hat{\mathbf{z}}, \hat{\hat{\mathbf{z}}})$) that are connected with vectors $\hat{\mathbf{a}}$ and $\hat{\hat{\mathbf{a}}}$. Both vectors $\hat{\mathbf{z}}$ and $\hat{\hat{\mathbf{z}}}$ has zero on the position where the original sets $\hat{\mathbf{a}}$ and $\hat{\hat{\mathbf{a}}}$ have real cross-section and one on the position of the virtual section $\bar{a}$. Those additional constraints satisfy that the virtual cross-sections are never assigned to the solution.

The same problem is possible to solve on benchmarks with bars assigned into groups as is already mentioned in Section 4. Cross-sectional areas are then chosen from different sets. It is necessary to realize that multiplication from the right hand side by the matrix $\mathbf{G}$ decreases a bar expression into a group expression, i.e. $\mathbf{\Psi} \cdot \mathbf{G}$ in which matrix $\mathbf{\Psi}$ is any matrix with proper size. From the other point of view, the extension from groups into rods is made through the multiplication from the left by the matrix $\mathbf{G}$, i.e. $\mathbf{G} \cdot \phi$ in which vector ϕ has a size $ng \cdot na$. Thus the prior approach with different sets is applicable to equations (21) to (27).

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \rho\Big((\boldsymbol{\ell} \otimes \mathbf{1}_{[na]})^T \odot (\mathbf{V} \otimes \mathbf{I}_{[na]}\mathbf{a})^T\Big)\, \mathbf{Gx} \tag{34}$$

$$\text{s.t. } \mathbf{Bs} = \mathbf{f}, \tag{35}$$

$$\sigma^{\min} \operatorname{diag}\big(\mathbf{V} \otimes \mathbf{I}_{[na]}\mathbf{a}\big)\, \mathbf{Gx} \leq \mathbf{s} \leq \sigma^{\max} \operatorname{diag}\big(\mathbf{V} \otimes \mathbf{I}_{[na]}\mathbf{a}\big)\, \mathbf{Gx}, \tag{36}$$

$$\Big(\big(\xi \otimes \mathbf{1}_{[na]} \odot (\mathbf{V} \otimes \mathbf{I}_{[na]}\mathbf{a}) \otimes \mathbf{1}^T_{[nd]}\big) \odot \mathbf{B}^T\Big)\mathbf{u} - \mathbf{s} \geq \big(\mathbf{1}_{[na\cdot nr]} - \mathbf{Gx}\big) \odot \mathbf{C}^{\min}, \tag{37}$$

$$\Big(\big(\xi \otimes \mathbf{1}_{[na]} \odot (\mathbf{V} \otimes \mathbf{I}_{[na]}\mathbf{a}) \otimes \mathbf{1}^T_{[nd]}\big) \odot \mathbf{B}^T\Big)\mathbf{u} - \mathbf{s} \leq \big(\mathbf{1}_{[na\cdot nr]} - \mathbf{Gx}\big) \odot \mathbf{C}^{\max}, \tag{38}$$

$$\mathbf{I}_{[nr]} \otimes \mathbf{1}_{[na]} \cdot \mathbf{Gx} = \mathbf{1}_{[nr]}, \tag{39}$$

$$\operatorname{diag}\big(\mathbf{V} \otimes \mathbf{I}_{[na]} \cdot \mathbf{z}\big) \cdot \mathbf{Gx} = \mathbf{0}_{[na\times nr]}, \tag{40}$$

$$\operatorname{diag}\big(\mathbf{V} \otimes \mathbf{I}_{[na]} \cdot \mathbf{z}\big) \cdot \mathbf{s} = \mathbf{0}_{[na\times nr]}, \tag{41}$$

$$\mathbf{x} \in \{0, 1\}. \tag{42}$$

5. Relaxation with cables

The difference for cables and ordinary bars is in carrying the stresses. The regular bars can be both in tension and compression whereas the cables cannot carry compression. The problem for tensile members and compressed bars remains the same as it was mentioned in Equations (6) to (13). The case with compressed cables is slightly different. A cable cannot be compressed thus $\sigma_{\min} = 0$. Therefore, the stress constraint (8) is adjusted to

$$0 \leq s_{i,j} \leq x_{i,j} a_i \sigma^{\max} \quad \forall (i, j). \tag{43}$$

Now, the compatibility condition is fulfilled only for the compression part, i.e. Equation (10). The term $(E_j a_i/\ell_j)\mathbf{b}_j^{\mathrm{T}}\mathbf{u}$ in Equation (9) can attain any negative value, i.e. in case of a bar, it will be in compression, in case of a cable, its sagging produces a zero stress. Therefore, the term $(1 - x_{i,j})c^{\min}_{i,j}$ in Equation (9) is replaced with minus infinity to enable any contraction of end nodes of the cable.

At this point we will limit our attention only to prescribed topology, i.e. to the pre-defined positions of bars and cables. A vector $\mathscr{M}$ differentiates whether the truss-member is a cable or a bar. For instance, a vector $[01100]$ defines a 1,4,5-bar structure stiffened with cables in positions 2 and 3; i.e. in case of a cable, the jth element is equal to one, otherwise it is equal to zero. Vector $\neg\mathscr{M}$ is a logical complement to vector $\mathscr{M}$. Next, the vector $\mathscr{M}$ is used to differentiate the limits of inequality (9). Then, the final derivation reads

$$\min_{\substack{\mathbf{x}\in\mathbb{B}^{na\cdot nr},\\ \mathbf{u}\in\mathbb{R}^{nd},\\ \mathbf{s}\in\mathbb{R}^{na\cdot nr}}} \quad \rho\Big((\boldsymbol{\ell}\otimes\mathbf{1}_{[na]})^T\odot(\mathbf{1}_{[nr]}\otimes\mathbf{a})^T\Big)\,\mathbf{x} \tag{44}$$

$$\text{s.t.} \quad \mathbf{B}\mathbf{s}=\mathbf{f}, \tag{45}$$

$$\sigma^{\min}\,\mathrm{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{x}\le\mathbf{s}\le\sigma^{\max}\,\mathrm{diag}\big(\mathbf{1}_{[nr]}\otimes\mathbf{a}\big)\,\mathbf{x}, \tag{46}$$

$$\begin{aligned}\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}^T_{[nd]}\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\ge{}&\mathrm{diag}\big(\mathscr{M}\otimes\mathbf{1}_{[na]}\big)\cdot(-\infty)+\dots\\ &+\mathrm{diag}\big(\neg\mathscr{M}\otimes\mathbf{1}_{[na]}\big)\odot\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{x}\big)\odot\mathbf{C}^{\min},\end{aligned} \tag{47}$$

$$\Big(\big(\xi\otimes\mathbf{a}\otimes\mathbf{1}^T_{[nd]}\big)\odot\mathbf{B}^T\Big)\mathbf{u}-\mathbf{s}\le\big(\mathbf{1}_{[na\cdot nr]}-\mathbf{x}\big)\odot\mathbf{C}^{\max}, \tag{48}$$

$$\mathbf{I}_{[nr]}\otimes\mathbf{1}_{[na]}\cdot\mathbf{x}=\mathbf{1}_{[nr]}, \tag{49}$$

$$\mathbf{x}\in\{0,1\}. \tag{50}$$

Finally, note that inclusion of groups as presented in Section 4. to the cable-truss formulation presented above is straightforward.

6. Conclusions

The presented contribution shows major steps needed for the relaxation of the cable-truss sizing optimization problem. At this point, a Branch and Bound method can be used. However, two issues remain unsolved. The first one is a free selection of a cross-sectional type, i.e. whether the rod will be a cable or a truss. This is a domain of topology optimization and therefore was not taken into account in our work. For more details, inspect e.g. a work on tensegrity structures Kanno (2011), which are a special part of cable-truss structures. Note, that the inclusion of the cross-sectional type needs additional binary variable for each rod which can dramatically complicate computational demands of the optimization task. The second issue deals with the prestressing of the cables. At this point, the procedure is relatively simple and is also presented in Kanno (2011).

7. Acknowledgment

The authors gratefully acknowledge the financial support from the Grant Agency of the Czech Republic through the project GAČR P105/12/1146 and from the Grant Agency of the Czech Technical University in Prague, the grant SGS13/034/OHK1/1T/11.

8. References

Arora, J. S. 2002: Methods for discrete variable structural optimization. In *Recent Advantages in Optimal Structural Design*, chapter 1, pages 1–40. American Society of Civil Engineers.

Bendsøe, M. P. 1995: *Optimization of structural topology, shape and material*. Springer-Verlag, 1 edition, 1995.

Fox, R. L. & Schmit, L. A. 1966: Advances in the integrated approach to structural synthesis. *Journal of Spacecraft and Rockets*, 3(6):858–866.

Kanno, Y. 2011: Topology optimization of tensegrity structures under compliance constraint: a mixed integer linear programming approach. *Optimization and Engineering*, pages 1–36, 2011.

Mitchell, J. E.. 2002: Branch-and-cut algorithms for combinatorial optimization problems. In *Handbook of Applied Optimization*, pages 65–77. Oxford University Press.

Pospíšilová, A. 2011: Search for global optima of sizing optimization benchmarks. Master's thesis, CTU in Prague, 2011. In Czech.
http://klobouk.fsv.cvut.cz/~pospisilova/publications/AP_DP_2011.pdf.

Pospíšilová, A. & Lepš, M. 2012: Global optima for size optimization benchmarks by branch and bound principles. *Acta Polytechnica*, 52(6):74–81.

Pospíšilová, A. & Lepš, M. 2013: Parallel Branch and Bound Method for Size Optimization Benchmarks- In B. H. V. Topping and P. Iványi, editors, *Proceedings of the Third International Conference on Parallel, Distributed, Grid and Cloud Computing for Engineering*, Stirlingshire, United Kingdom, 2013. Civil-Comp Press. paper 20.

Rasmussen, M. H. & Stolpe, M. 2008: Global optimization of discrete truss topology design problems using a parallel cut-and-branch method. *Computers & Structures*, 86:1527–1538.

Wu, S.-J. & Chow, P.-T. 1995: Steady-state genetic algorithms for discrete optimization of trusses. *Computers & Structures*, 56(6):979–991.

A Matrix products

Hadamard product (also known as the element-wise product or dot product) of matrices $\mathbf{A}_{[n\times p]} = (a_{i,j})$ and $\mathbf{B}_{[n\times p]} = (b_{i,j})$ is the matrix $\mathbf{C}_{[n\times p]} = c_{i,j} = a_{i,j}b_{i,j}$. Matrices $\mathbf{A}$ and $\mathbf{B}$ has to have same size. The notation is $\mathbf{C} = \mathbf{A} \odot \mathbf{B}$.

$$\mathbf{C} = \mathbf{A} \odot \mathbf{B} = (a_{i,j}b_{i,j}) = \begin{bmatrix} a_{1,1}b_{1,1} & \cdots & a_{1,p}b_{1,p} \\ \vdots & & \vdots \\ a_{n,1}b_{n,1} & \cdots & a_{n,p}b_{n,p} \end{bmatrix}. \tag{51}$$

Kronecker product (also known as the tensor product) of matrices $\mathbf{A}_{[n\times p]} = (a_{i,j})$ and $\mathbf{D}_{[m\times q]} = (d_{k,l})$ is the matrix $\mathbf{E}$ with size $mn \times pq$ and defined as

$$\mathbf{E} \;=\; \mathbf{A} \otimes \mathbf{D} = (a_{i,j}\mathbf{D}) = \begin{bmatrix} a_{1,1}\mathbf{D} & \cdots & a_{1,p}\mathbf{D} \\ \vdots & & \vdots \\ a_{n,1}\mathbf{D} & \cdots & a_{n,p}\mathbf{D} \end{bmatrix}. \tag{52}$$