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DYNAMIC RESPONSE OF A HEAVY BALL ROLLING INSIDE A SPHERICAL DISH UNDER EXTERNAL EXCITATION

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Summary: *The set of a heavy metallic ball which is rolling freely inside a semi-spherical dish with larger diameter, being fixed to structure, is frequently used as tuned mass damper of vibration. Ratio of both diameters, mass of the rolling ball, quality of contact surfaces and other parameters should correspond with characteristics of the structure. The ball damper is modelled as a non-holonomic system. Hamiltonian functional including an adequate form of the Rayleigh function is formulated in moving coordinates using Euler angles and completed by ancillary constraints via Lagrangian multipliers. Subsequently Lagrangian differential system is carried out. Together with rolling conditions the governing system of seven equations is formulated. Later Lagrangian multipliers character is analysed and redundant motion components are eliminated. First integrals are derived and main energy balances evaluated together with their physical interpretation. Discussion of basic dynamic properties of the system is provided.*

Keywords: ***Non-holonomic systems, Hamilton functional with constrains, Moving coordinates, Non-linear vibration, Vibration ball absorber***

1. Introduction

Passive vibration absorbers of various types are very widely used in civil engineering, especially when wind induced vibration should be suppressed. TV towers, masts and other slender structures exposed to wind excitation are usually equipped by such devices. Conventional passive absorbers are of the pendulum type, see e.g. (Haxton & Barr, 1974), where auto-parametric type absorber is described. Although they are very effective and reliable, they have several disadvantages limiting their application. First of all, they have certain requirements to space, particularly in a vertical direction. These requirements cannot be satisfied any time when an absorber should be installed as a supplementary equipment. Also horizontal construction, like foot bridges, cannot accept any absorber of the pendulum type. Another disadvantage represents a need of a regular maintenance.

Both above shortcomings can be avoided using the absorber of ball type. The basic principle comes out of a rolling movement of a metallic ball of a radius r inside of a metallic rubber coated spherical dish of a radius $R > r$, Fig. 1. This system is closed in an airtight case. Such a device is practically maintenance free. Its vertical dimension is relatively very small and can

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be used also in such cases where a pendulum absorber is inapplicable due to lack of vertical space or difficult maintenance. First papers dealing with the theory and practical aspects of ball absorbers have been published during the last two decades, see Pirner (1994) and Pirner & Fischer (2000). The first paper dealing with the problem on the basis of the rational dynamics has been published some years ago, see Náprstek & Pirner (2002) and Náprstek et al. (2011). The above referenced papers are probably the first attempts to present basic mathematical model in 2D together with its numerical evaluation and practical application as far as to the state of the realization including some results of long-term in situ measurements.

Dynamics of the real ball absorber is more complicated in comparison with the pendulum one. Its movement can be hardly described in a linear state although for the first view its behaviour is similar to the pendulum absorber type. A number of problems are still open being related with movement stability, bifurcations, auto-parametric resonances, see e.g. Nabergoj & Tondl (1994), Tondl (1997), and at least but not last originating from the spherical dish and ball surface imperfections. The ball moving inside the spherical dish is very sensitive to the stability loss of the semi-trivial state representing the movement in a vertical plane. However this type of the ball motion is requested, as it provides the optimal efficiency of any damper, see e.g. Lee & Hsu (1994) or Náprstek & Fischer (2009). Therefore any stability loss of the semi-trivial state deteriorates or invalidates any effect of the device. Due to probability of the stability loss, which is much higher than of the spherical pendulum, semi-trivial states should be carefully analysed including a large variety of post-critical processes. Experiences with other auto-parametric systems prone to the physical stability loss warn about various types of numerical stability loss during simulations. So this factor should be taken into account since the basic formulation of the mathematical model, see e.g. Rosenstein et al. (1993) or Ren & Beards (1994). It means that domains including boundaries of transmission into the post-critical states and backwards should be subdued to special dealing making possible to define relevant limits.

Figure 1: Outline of the ball vibration absorber; left: basic scheme; right: photo of a full scale device

2. Mathematical background

The slipping-less movement of a ball on a surface is a non-holonomic problem. So constraints relating mutual movement of a ball and a surface include velocity components of their movement. Putting together expression for kinetic and potential energies T, V, Rayleigh function Υ and external forces $\mathbf{Q}$, the relevant Lagrangian equations should be written in a form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial \Upsilon}{\partial \dot{q}_j} = Q_j + \sum_{m=1}^{l} \lambda_m \cdot B_{mj}, \qquad j = 1, ..., n \tag{1}$$

together with non-holonomic constraints:

$$\sum_{j=1}^{n} B_{mj} \cdot \dot{q}_j + B_m = 0. \qquad m = 1, ..., l. \tag{2}$$

where q_j $(j = 1, ..., n)$ are generalized coordinates. Symbols B_{mj} and B_m are generally functions of q_j. Explicit time can be usually omitted as constraints are considered skleronomous as a rule if external kinematic excitation is absent and only initial conditions provide energy to the system. Provided kinematic excitation works, then respective parameters $B_m \neq 0$ and should be considered as functions of time. Symbols λ_m $(m = 1, ..., l)$ are Lagrangian multiplicators being used to add non-holonomic conditions to the basic Hamiltonian functional. Therefore the system (1), (2) includes $n + l$ unknowns q_j, λ_m, see for instance popular monographs Arnold (1978) or Hamel (1978). Constraints Eqs (10) are formulated as linear functions of velocities $\dot{q}_j$ which reveals to be satisfactory concerning problems considered. However non-linear non-holonomic condition are also applied in some cases, see for instance Hamel (1978).

Let us pay attention to quantities mentioned above. Basically they should be expressed in the first step following moving coordinates, because in this system can be easily introduced contact conditions and moreover processes inside the system can be better illustrated using moving coordinates. The origin of moving coordinates is located in the center of the moving ball, so that following the concentric sphere with the dish of the diameter $\varrho = R - r$. Moving axis $1, 2, 3$ follows a tangent of the concentric sphere meridian in a vertical plane ξ, z, axis 2 is always horizontal and axis 3 follows a normal to the tangential plane in contact of both bodies being directed upwards, see Fig. 2 - left picture: axonometric view and right picture: some details in vertical plane ξ, z. Components of velocity vector $\boldsymbol{\omega}$ in moving coordinates $\omega_1, \omega_2, \omega_3$ are positive correspondingly with usual convention. Angles α, γ determine position of the contact point.

Basic formulae for kinetic and potential energies with respect to moving coordinates read:

$$\begin{aligned} T &= \tfrac{1}{2} m[v_1^2 + v_2^2 + v_3^2 + \tfrac{2}{5} r^2(\omega_1^2 + \omega_2^2 + \omega_3^2)] && \text{(a)} \\ V &= mg\varrho(1 - \cos\alpha) \qquad (\varrho = R - r) && \text{(b)} \end{aligned} \tag{3}$$

Rayleigh function is introduced in a form leading to linear viscous damping in individual moving coordinates. In particular separately in $1, 2$ axes parallel with the tangential plane in contact point of the ball and the dish quantified by components ω_1, ω_2 and in axis 3 being normal to the contact plane with velocity ω_3:

$$\Upsilon = \frac{1}{2}\delta_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}\delta_2\omega_3^2 \tag{4}$$

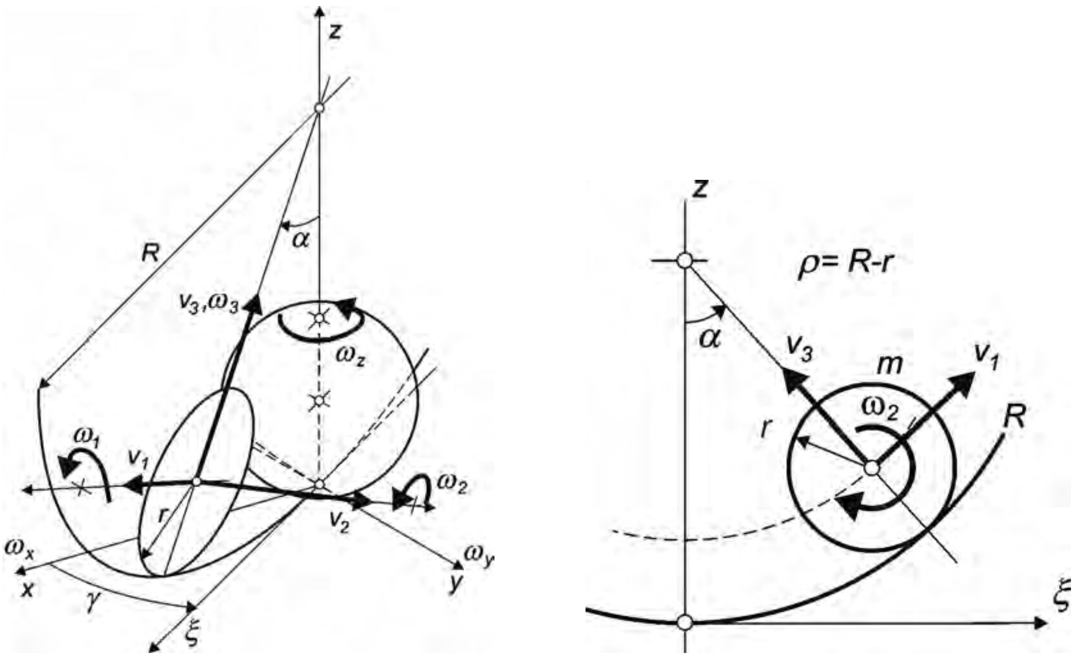

Figure 2: Outline of coordinate systems; left: axonometric view; right: plane ξz view - along γ orientation

This formula indicates that only the rolling resistance and rotation friction proportional to relevant velocity components are respected. No axial damping forces proportional to air flow velocity are taken into account. Quantification of rolling resistance is specified by δ_1 parameter while friction related with rotation around 3 axis is considered to be δ_2. Generally it holds $\delta_1 \neq \delta_2$. How far is justified this simple model of energy dissipation will be investigated in a separate study.

In order to put a relation of velocity vector $\boldsymbol{\omega}$ projections into moving coordinates $\omega_1, \omega_2, \omega_3$ and those into fixed coordinates of the dish $\omega_x, \omega_y, \omega_z$, we write in compliance with Fig. 2:

$$\begin{array}{llllllll} \omega_x & = & & \omega_1 \cos\alpha\cos\gamma & - & \omega_2 \sin\gamma & + & \omega_3 \sin\alpha\cos\gamma \quad \text{(a)} \\ \omega_y & = & & \omega_1 \cos\alpha\sin\gamma & + & \omega_2 \cos\gamma & + & \omega_3 \sin\alpha\sin\gamma \quad \text{(b)} \\ \omega_z & = & - & \omega_1 \sin\alpha & & & & \omega_3 \cos\alpha \quad \text{(c)} \end{array} \tag{5}$$

In order to put a relation of velocity vector $\boldsymbol{\omega}$ projections into moving coordinates $\omega_1, \omega_2, \omega_3$ and velocity components in Euler angles φ, θ, ψ, we write:

$$\begin{array}{llllllll} \omega_1 & = & - & \dot\varphi \sin\theta\cos\psi & + & \dot\theta \sin\psi & & \quad \text{(a)} \\ \omega_2 & = & & \dot\varphi \sin\theta\sin\psi & + & \dot\theta \cos\psi & & \quad \text{(b)} \\ \omega_3 & = & & \dot\varphi \cos\theta & & & + & \dot\psi \quad \text{(c)} \end{array} \tag{6}$$

where it certainly holds for components of the vector $\boldsymbol{\omega}$: $\omega_x^2 + \omega_y^2 + \omega_z^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Considering Fig. 2, following contact relations reveal obvious:

$$v_1 \; = \; \varrho\dot\alpha, \quad v_2 \; = \; \varrho\dot\gamma\sin\alpha, \quad v_3 \; = \; 0, \qquad (\varrho = R - r) \tag{7}$$

where the expression $v_3 = 0$ represents one of the three contact constraints.

With reference to Eqs (6) and (7) the kinetic energy T, see Eq. (3a), can be formulated as a function of Euler angles velocities:

$$T = \frac{1}{2}m\left[\varrho^2(\dot{\alpha}^2 + \dot{\gamma}^2\sin^2\alpha) + \frac{2}{5}r^2(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta)\right] \tag{8}$$

Take a note that evaluating the kinetic energy the square of the absolute value $|\boldsymbol{\omega}|^2$ can be evaluated using either Eqs (5) or Eqs (6) with identical results.

Similarly the Rayleigh function, Eq. (4) can be rewritten as follows:

$$\Upsilon = \frac{1}{2}\delta_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}\delta_2(\dot{\psi}^2 + \dot{\varphi}^2\cos^2\theta + 2\dot{\varphi}\dot{\psi}\cos\theta) \tag{9}$$

Using first two expressions of Eqs (7), complete set of the contact constraints can be formulated:

$$\begin{array}{rcl} v_1 - r\omega_2 &=& 0, \\ v_2 + r\omega_1 &=& 0, \\ v_3 &=& 0 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} r\omega_2 - \varrho\dot{\alpha} &=& 0, \\ r\omega_1 + \varrho\dot{\gamma}\sin\alpha &=& 0. \\ v_3 &=& 0 \end{array} \tag{10}$$

Considering Eqs (7), first two constraints Eqs (10) can be rewritten in a form:

$$\begin{array}{rcll} r[\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi] - \varrho\dot{\alpha} &=& 0, & \text{(a)} \\ r[\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi] + \varrho\dot{\gamma}\sin\alpha &=& 0. & \text{(b)} \end{array} \tag{11}$$

Therefore we have two non-holonomic constraints Eqs (11a,b) and the matrix $\mathbf{B}$ appearing in Eqs (1) and (2) can be defined:

$$\mathbf{B} = \begin{bmatrix} -\varrho, & 0 \\ 0, & \varrho\sin\alpha \\ r\sin\theta\sin\psi, & -r\sin\theta\cos\psi \\ r\cos\psi, & r\sin\psi \\ 0, & 0 \end{bmatrix}^T \tag{12}$$

where a following association with symbols q_j in Eq. (2) has been adopted: $[\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5] = [\dot{\alpha}, \dot{\gamma}, \dot{\varphi}, \dot{\theta}, \dot{\psi}]$.

Let us collect the partial expressions occurring in the Lagrange system Eq. (1) being given by Eqs (3a), (8), (9), (11) and (12). Consider forces $Q_j = 0$, as the ball is not excited by external forces. Hence the Lagrange governing differential system can be carried out:

$$\begin{array}{rcllr} \varrho\ddot{\alpha} - \varrho\dot{\gamma}^2\sin\alpha\cos\alpha + g\sin\alpha &=& -\ \lambda_1 & & \text{(a)} \\ \varrho(\ddot{\gamma}\sin\alpha + 2\dot{\gamma}\dot{\alpha}\cos\alpha)\sin\alpha &=& & \lambda_2\sin\alpha & \text{(b)} \\ \frac{2}{5}r(\ddot{\varphi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) + & & & & \\ \delta_1\dot{\varphi}\sin^2\theta + \delta_2(\dot{\varphi}\cos^2\theta + \dot{\psi}\cos\theta) &=& \lambda_1\sin\theta\sin\psi & -\lambda_2\sin\theta\cos\psi & \text{(c)} \\ \frac{2}{5}r(\ddot{\theta} + \dot{\psi}\dot{\varphi}\sin\theta) + \delta_1\dot{\theta} &=& \lambda_1\cos\psi & +\lambda_2\sin\psi & \text{(d)} \\ \ddot{\psi} + \ddot{\varphi}\cos\theta - \dot{\varphi}\dot{\theta}\sin\theta + & & & & \\ \delta_2(\dot{\varphi}\cos\theta + \dot{\psi}) &=& 0 & & \text{(e)} \end{array} \tag{13}$$

The system of Eqs (13) together with constraints Eqs (11) contains seven unknown response components: $[\alpha, \gamma, \varphi, \theta, \psi, \lambda_1, \lambda_2]$ and hence it represents a closed governing differential system.

Lagrange multipliers λ_1, λ_2 can be interpreted as tangential reactions (divided by mass m of a ball) in the contact point in respective moving axes. They can be eliminated. Using Eqs (13c,d) one obtains:

$$
\begin{aligned}
\lambda_1 \sin\theta \;&=\; [\tfrac{2}{5} r \tfrac{\mathrm{d}}{\mathrm{d}t}(\dot\varphi + \dot\psi\cos\theta) + \delta_1\dot\varphi\sin^2\theta + \delta_2\omega_3\cos\theta]\sin\psi + \\
&\qquad [\tfrac{2}{5}(\ddot\theta + \dot\psi\dot\varphi\sin\theta) + \delta_1\dot\theta]\sin\theta\cos\psi \qquad \text{(a)} \\
\lambda_2 \sin\theta \;&=\; -[\tfrac{2}{5} r \tfrac{\mathrm{d}}{\mathrm{d}t}(\dot\varphi + \dot\psi\cos\theta) + \delta_1\dot\varphi\sin^2\theta + \delta_2\omega_3\cos\theta]\cos\psi + \\
&\qquad [\tfrac{2}{5}(\ddot\theta + \dot\psi\dot\varphi\sin\theta) + \delta_1\dot\theta]\sin\theta\sin\psi \qquad \text{(b)}
\end{aligned}
\tag{14}
$$

so the reduced form of the system Eqs (13) reads:

$$
\begin{aligned}
(\varrho\ddot\alpha - \varrho\dot\gamma^2\sin\alpha\cos\alpha + g\sin\alpha)\sin\theta \;&=\; -[\tfrac{2}{5} r(\ddot\theta + \dot\psi\dot\varphi\sin\theta) + \delta_1\dot\theta]\sin\theta\cos\psi - \\
&\quad [\tfrac{2}{5} r \tfrac{\mathrm{d}}{\mathrm{d}t}(\dot\varphi + \dot\psi\cos\theta) + \delta_1\dot\varphi\sin^2\theta + \delta_2(\dot\psi + \dot\varphi\cos\theta)\cos\theta]\sin\psi \qquad \text{(a)} \\
(\varrho(\ddot\gamma\sin\alpha + 2\dot\gamma\dot\alpha\cos\alpha)\sin\alpha)\sin\theta \;&=\; [\tfrac{2}{5} r(\ddot\theta + \dot\psi\dot\varphi\sin\theta) + \delta_1\dot\theta]\sin\theta\sin\psi\sin\alpha - \\
&\quad [\tfrac{2}{5} r \tfrac{\mathrm{d}}{\mathrm{d}t}(\dot\varphi + \dot\psi\cos\theta) + \delta_1\dot\varphi\sin^2\theta + \delta_2(\dot\psi + \dot\varphi\cos\theta)\cos\theta]\cos\psi\sin\alpha \qquad \text{(b)} \\
(\tfrac{\mathrm{d}}{\mathrm{d}t} + \delta_2)(\dot\psi + \dot\varphi\cos\theta) \;&=\; 0. \qquad \text{(c)}
\end{aligned}
\tag{15}
$$

Reduced system (15) together with constraints Eqs (11) makes the governing differential system for unknown components $\alpha, \gamma, \varphi, \theta, \psi$ of the dynamic system response.

Figure 3 shows sample solution of the system (15,11) for free vibration. Initial conditions were set up as $\alpha = 0.1, \gamma = 1$. It is obvious, that in this case is γ constant, value of α decays due to damping coefficient δ_1. Plots of the Euler angles φ, θ, ψ are shown in the second row of Figure 3.

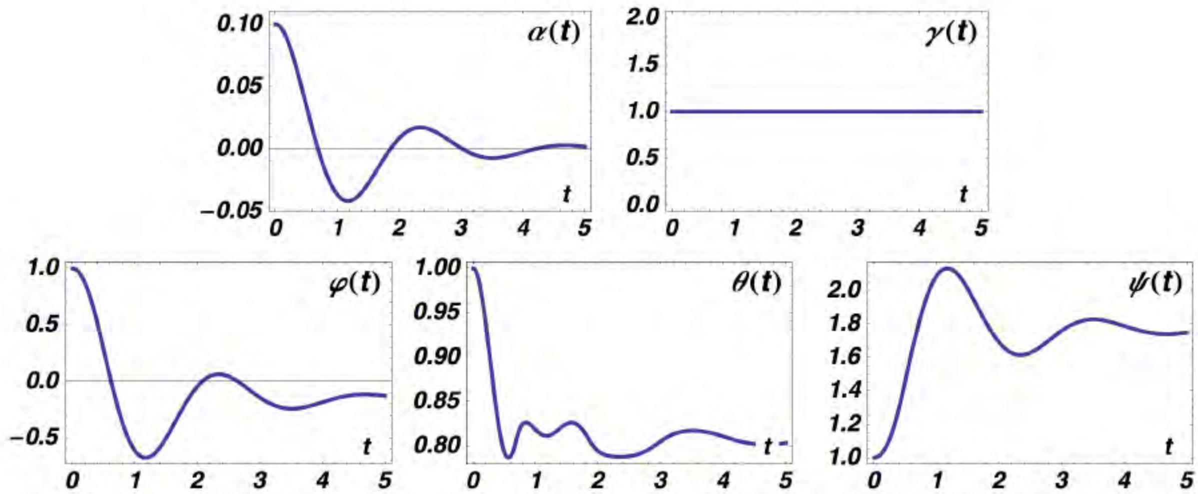

Figure 3: Numerical solution of (15,11) for $r = 0.1,\ \rho = 0.9,\ \delta_1 = 0.21,\ \delta_2 = 0.22$.

3. Special cases

It is worthy to compare mathematical models of several simpler cases with the general system Eqs (15) and (11).

Let us go through the case of the ball moving in a vertical plane. It means: $\gamma = \varphi = \psi = 0$ and subsequently also $\lambda_2 = 0$ and $\omega_3 = 0$ as it follows from (14b) and (18). Then Eqs (15a) and (11a) gain a simplified form:

$$\begin{array}{rcll} \varrho\ddot{\alpha} + g\sin\alpha & = & -(\frac{2}{5}r\ddot{\theta} + \delta_1\dot{\theta}) & \text{(a)} \\ r\dot{\theta} - \varrho\dot{\alpha} & = & 0 & \text{(b)} \end{array} \tag{16}$$

and so it holds:

$$\ddot{\alpha} + \frac{5}{7}\frac{\delta_1}{\varrho}\dot{\alpha} + \frac{5}{7}\frac{g}{\varrho}\sin\alpha = 0 \tag{17}$$

Equation (17) can be found in various references, e.g. Náprstek & Pirner (2002). It represents the mathematical pendulum with rotating mass following the contact constraint. Rotation of the mass increases an effective mass value due to inertia moment of the ball.

The second demonstration follows from the fact that Eq. (13e) is independent from Lagrange multipliers. With respect to Eq. (6c) it can be rewritten in the form:

$$(\frac{\mathrm{d}}{\mathrm{d}t} + \delta_2)(\dot{\psi} + \dot{\varphi}\cos\theta) \;=\; \frac{\mathrm{d}\omega_3}{\mathrm{d}t} + \delta_2\omega_3 \;=\; 0 \tag{18}$$

and hence

$$\omega_3 \;=\; \omega_s \exp(-\delta_2 t) \tag{19}$$

where ω_s is an arbitrary constant representing an initial velocity introduced independently. Formula Eq. (19) represents the first integral of the ball spin around the ω_3 axis. The ball rotation around the axis perpendicular to the dish is independent from remaining components of the system response. Velocity ω_3 is exponentially dropping with time if $\delta_2 > 0$, otherwise remains constant.

As a third example we examine the ball movement excluding the contact of the ball and dish. So any rotation of the ball is eliminated. Hence Euler angles φ, θ, ψ vanish and so they are $\lambda_1 = \lambda_2 = 0$. Consequently only homogenized Eqs (13a,b) remain in force. They can be written as follows:

$$\begin{array}{rcll} \ddot{\alpha} - \dot{\gamma}^2\sin\alpha\cos\alpha + \frac{g}{\varrho}\sin\alpha & = & 0 & \text{(a)} \\ \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\gamma}\sin^2\alpha) & = & 0 & \text{(b)} \end{array} \tag{20}$$

comprising governing system of a spherical pendulum with a stiff suspension. For details see many references, eg. Hamel (1978) or Náprstek & Fischer (2009). For a certain constant value of $\dot{\gamma} = \omega_r$ the ball moves in a horizontal plane. In such a case $\alpha = \alpha_r$ and Eq. (13a) can be rewritten in a form:

$$\omega_r^2\cos\alpha_r - \omega_0^2 - 0 \implies \cos\alpha_r = \frac{\omega_0^2}{\omega_r^2}; \quad ; \qquad (\omega_0^2 = \frac{g}{\varrho}) \tag{21}$$

where should be valid $\omega_r > \omega_0$ on order to keep stability of the regime.

4. Conclusions

The mathematical model of the ball type vibration absorber has been outlined. Movement of the heavy ball in a spherical dish has been investigated. With respect to a usual geometry it has been shown that this device is much more sensitive to the stability loss of the semi-trivial solution in a vertical plane. Preliminary theoretical and mainly experimental investigation revealed that the non-linear character of this device is an important factor influencing significantly its dynamic properties and practical efficiency. Simplification leading to various types of linearised models is hardly acceptable. Moreover, it seems that the non-linear character making the form of resonance curves dependent on the excitation amplitude leads to better efficiency in comparison with linear mechanism. Approximation of strong non-linearity by means of polynomial approaches widely used in theoretical studies of spherical pendulum response are not possible as well. At least the first qualitative analyses of solution types, resonance zones, transmissions between regular and chaotic regimes should respect the strong non-linearity of the system.

The basic Lagrangian analytical theory of non-linear behaviour has been done. The model has been approached as non-holonomic with five degrees of freedom completed by two non-trivial reaction components in a form of Lagrange multipliers. Energy dissipation has been introduced via Rayleigh function considering linear dependence of damping forces on angular velocity components of the ball in moving coordinates. Several special cases have been investigated in order to compare the model developed with conventional partial models investigated earlier. On the other hand some other types of mathematical models are worthy to be composed, as the way using the strict Lagrangian strategy doesn't proved a high efficiency from the viewpoint of further investigation, its transparency and physical interpretation possibility. At least kinematic approach and Appels-Gibbs theory should be investigated before large scale simulation program will be started.

Laboratory tests of the vibration ball absorber with the dish without and with rubber coating have demonstrated several aspects of real operation of the damper. With respect to laboratory tests and long-term in situ measurements can be concluded that the vibration ball absorber is a simple nearly maintenance free low cost device with very small vertical dimensions. For these properties it is very convenient for application especially in cases when broad band excitation of random character prevails and when very limited vertical space is available.

5. Acknowledgement

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