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FREE VIBRATION ANALYSIS OF EULER-BERNOULLI BEAM WITH DISCONTINUITIES BY MEANS OF DISTRIBUTIONS

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Abstract: *The general equation for the transverse vibration of Euler-Bernoulli beam has been used since it was derived by means of classical derivatives of the shear force, the bending moment, the slope and the deflection of the beam. However these derivatives are not defined at such points of a center-line between ends of the beam in which there is a concentrated support or a concentrated mass or a concentrated mass moment of inertia or an internal hinge connecting beam segments, which are discontinuities that can be met in practice. We have applied the distributional derivative for a discontinuous shear force, a discontinuous bending moment, and a discontinuous slope of the beam in order to derive generalized mathematical model for free transverse vibration as a system of partial differential equations. We have computed general solution to the generalized mathematical model for prismatic beam by means of a symbolic programming approach via Maple. As a result of this approach, computing natural frequencies and modal shapes of a slender beam, it is not necessary for continuity conditions to be put together at discontinuity points mentioned. As an example we have used this approach to obtain a frequency equation of a beam on three pin rigid supports, which would be more complicated if we tried to apply the transfer matrix method.*

Keywords: ***Euler-Bernoulli beam, transverse vibration, discontinuities, Dirac distribution*.**

1. Introduction

Classical analytical method of calculating natural frequencies of a beam with discontinuities is based on the following main steps (Timoshenko, 1937). Firstly we divide the beam into segments without discontinuities. Secondly we find continuous solution to a differential equation of motion for each segment separately. Thirdly we express boundary conditions for each segment, and continuity conditions among adjoining segments leading to a homogeneous system of linear algebraic equations. Finally we derive a frequency equation as a condition of nontrivial solution to the homogeneous system of linear equations.

Applying the distributional derivative definition for a discontinuous shear force, a discontinuous bending moment, and a discontinuous slope of a beam, we can derive a mathematical model for free transverse vibration of Euler-Bernoulli beam with discontinuities caused by concentrated supports or concentrated masses or concentrated mass moments of inertia situated between ends of the beam, or hinges connecting beam segments. This mathematical model can be solved like only one differential task without dividing the beam into segments where all the continuity conditions among adjoining segments are fulfilled automatically. Using this approach, we always have only four integration constants irrespective of the number of the discontinuities.

2. The classical equation of motion for free transverse vibration of Euler-Bernoulli beam

Neglecting the effects of rotary inertia and shear deformation, and supposing no axial loading of the slender beam, we may express the equation of motion for free transverse vibration of the beam without discontinuities in the shear force, the bending moment, the slope and the deflection (Rao, 2007) as

$$\rho\ \mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right) = -E\left(\frac{\partial^2}{\partial x^2}\left(\mathrm{J}(x)\left(\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)\right)\right)\right) \quad , \tag{1}$$

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where w(x,t) is the deflection, ρ is the density of the beam material, A(x) is the cross-sectional area, E is the modulus of elasticity (Young's modulus), and J(x) is the area moment of inertia with respect to the centroid axis which is perpendicular to the plane of vibration of the beam.

3. A mathematical model for free transverse vibration of Euler-Bernoulli beam with discontinuities

Expressing the first classical derivative of the shear force (Q) with respect to x from the equation of motion for an element cut out of the beam (Juliš at al., 1987; Brepta at al., 1994), and adding discontinuities of this force by using the distributional derivative definition (Schwartz, 1972; Štěpánek, 2001; Kanwal, 2004), we can derive Eq. (2) in which $r_i(t)$ is a reaction at ith concentrated support at a point $x=a_i$ ($0< a_i < l$), l is the length of the beam, m_i is a concentrated mass at a point $x=b_i$ ($0< b_i <l$), and Dirac(x-a_i) denotes the Dirac distribution.

When a beam carrying concentrated masses with moments of inertia J_i at points $x=b_i$ is vibrating, jump discontinuities in the bending moment may occur at these points. Expressing the first classical derivative of the bending moment (M) from the static equilibrium equation for an element cut out of the beam, and adding discontinuities of this moment multiplied by the Dirac distribution moved to the points of the discontinuities, we can obtain Eq. (3).

If a beam contains hinges connecting segments of the beam at points $x=c_i$ ($0< c_i < l$), discontinuities in the slope (ϕ) of magnitude $\psi_i(t)$ may be found at these points. Expressing the first classical derivative of the slope from the relation between the bending moment and the beam centerline curvature, and adding distributional parts containing $\psi_i(t)$ to the classical part of the distributional derivative, we can acquire Eq. (4).

$$\frac{\partial}{\partial x}\mathrm{Q}(x,t) = \rho\,\mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right) + \left(\sum_{i=1}^{n_1} r_i(t)\,\mathrm{Dirac}(x-a_i)\right) + \left(\sum_{i=1}^{n_2} m_i\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)\bigg|_{x=b_i}\mathrm{Dirac}(x-b_i)\right) \quad , \qquad (2)$$

$$\frac{\partial}{\partial x}\mathrm{M}(x,t) = \mathrm{Q}(x,t) - \left(\sum_{i=1}^{n_2} J_i\left(\frac{\partial^2}{\partial t^2}\phi(x,t)\right)\bigg|_{x=b_i}\mathrm{Dirac}(x-b_i)\right) \quad , \qquad (3)$$

$$\frac{\partial}{\partial x}\phi(x,t) = -\frac{\mathrm{M}(x,t)}{E\,\mathrm{J}(x)} + \left(\sum_{i=1}^{n_3}\psi_i(t)\,\mathrm{Dirac}(x-c_i)\right) \quad , \qquad (4)$$

$$\frac{\partial}{\partial x}\mathrm{w}(x,t) = \phi(x,t) \quad . \qquad (5)$$

4. Free vibration solution

Supposing a form of the solution to equations (2) to (5) as

$$\mathrm{Q}(x,t) = Q_s(x)\,\mathbf{e}^{(\Omega\,t\,I)} \quad , \qquad \mathrm{M}(x,t) = M_s(x)\,\mathbf{e}^{(\Omega\,t\,I)} \quad ,$$

$$\phi(x,t) = \phi_s(x)\,\mathbf{e}^{(\Omega\,t\,I)} \quad , \qquad \mathrm{w}(x,t) = w_s(x)\,\mathbf{e}^{(\Omega\,t\,I)} \quad ,$$

$$r_i(t) = R_i\,\mathbf{e}^{(\Omega\,t\,I)} \quad , \qquad \psi_i(t) = \Psi_i\,\mathbf{e}^{(\Omega\,t\,I)} \quad ,$$

where $I^2 = -1$,

and denoting: $\mathrm{W}_i = w_s\big|_{x=b_i}$, $\Phi_i = \phi_s\big|_{x=b_i}$,

we can derive ordinary differential equations (6) to (9) for unknown shapes of the deflection (w_s), the slope (ϕ_s), the bending moment (M_s), and the shear force (Q_s) as follows:

$$\frac{d}{dx} Q_s(x) = -\rho\, A\, w_s(x)\, \Omega^2 + \left(\sum_{i=1}^{n_1} R_i \text{ Dirac}(x - a_i) \right) - \left(\sum_{i=1}^{n_2} m_i\, W_i\, \Omega^2 \text{ Dirac}(x - b_i) \right) \quad , \tag{6}$$

$$\frac{d}{dx} M_s(x) = Q_s(x) + \left(\sum_{i=1}^{n_2} J_i\, \Phi_i\, \Omega^2 \text{ Dirac}(x - b_i) \right) \quad , \tag{7}$$

$$\frac{d}{dx} \phi_s(x) = -\frac{M_s(x)}{E\,J} + \left(\sum_{i=1}^{n_3} \Psi_i \text{ Dirac}(x - c_i) \right) \quad , \tag{8}$$

$$\frac{d}{dx} w_s(x) = \phi_s(x) \quad . \tag{9}$$

5. Characteristic functions of prismatic Euler-Bernoulli beam with discontinuities

In order to simplify expressions of the general solution to equations (6) to (9) for a prismatic beam, we introduce the notation:

$$\beta^4 = \frac{A\,\rho\,\Omega^2}{E\,J} \quad , \tag{10}$$

where Ω is a natural circular frequency of vibration.

We have used the Laplace transform method so as to compute general solution to the system of Eqs. (6)-(9), i.e. characteristic functions of the beam, with integration constants in the form of initial parameters:

$$\begin{aligned}
Q_s(x) = &\left(\frac{1}{2}\cos(\beta\, x) + \frac{1}{2}\cosh(\beta\, x) \right) Q_s(0) + \frac{1}{2}\left(\sinh(\beta\, x) - \sin(\beta\, x)\right) M_s(0)\, \beta \\
&+ \frac{1}{2}\, \frac{(\cos(\beta\, x) - \cosh(\beta\, x))\, A\, \rho\, \Omega^2\, \phi_s(0)}{\beta^2} - \frac{1}{2}\, \frac{(\sinh(\beta\, x) + \sin(\beta\, x))\, A\, \rho\, \Omega^2\, w_s(0)}{\beta} \\
&+ \left(\sum_{i=1}^{n_1} \frac{(\cosh(\beta\,(x - a_i)) + \cos(\beta\,(x - a_i)))\, \text{Heaviside}(x - a_i)\, R_i}{2} \right) \\
&- \left(\sum_{i=1}^{n_2} \frac{m_i\, (\cosh(\beta\,(x - b_i)) + \cos(\beta\,(x - b_i)))\, \text{Heaviside}(x - b_i)\, \Omega^2\, W_i}{2} \right) \\
&+ \frac{1}{2}\left(\sum_{i=1}^{n_2} (-\sin(\beta\,(x - b_i)) + \sinh(\beta\,(x - b_i)))\, J_i\, \beta\, \text{Heaviside}(x - b_i)\, \Omega^2\, \Phi_i \right) \\
&+ \frac{1}{2}\left(\sum_{i=1}^{n_3} \frac{\text{Heaviside}(x - c_i)\, \Psi_i\, (\cos(\beta\,(x - c_i)) - \cosh(\beta\,(x - c_i)))\, A\, \rho\, \Omega^2}{\beta^2} \right)
\end{aligned} \quad , \tag{11}$$

$$
\begin{aligned}
M_s(x) = {} & \frac{1}{2}\,\frac{(\sinh(\beta\,x)+\sin(\beta\,x))\,Q_s(0)}{\beta} + \left(\frac{1}{2}\cos(\beta\,x)+\frac{1}{2}\cosh(\beta\,x)\right) M_s(0) \\
& + \frac{1}{2}\,\frac{(\sin(\beta\,x)-\sinh(\beta\,x))\,\Omega^2\,\rho\,A\,\phi_s(0)}{\beta^3} + \frac{1}{2}\,\frac{(-\cosh(\beta\,x)+\cos(\beta\,x))\,\Omega^2\,\rho\,A\,w_s(0)}{\beta^2} \\
& + \frac{1}{2}\left(\sum_{i=1}^{n_1} \frac{\mathrm{Heaviside}(x-a_i)\,(\sin(\beta\,(x-a_i))+\sinh(\beta\,(x-a_i)))\,R_i}{\beta}\right) \\
& - \frac{1}{2}\left(\sum_{i=1}^{n_2} \frac{\Omega^2\,m_i\,(\sin(\beta\,(x-b_i))+\sinh(\beta\,(x-b_i)))\,\mathrm{Heaviside}(x-b_i)\,\mathrm{W}_i}{\beta}\right) \\
& + \frac{1}{2}\left(\sum_{i=1}^{n_3} \frac{\mathrm{Heaviside}(x-c_i)\,\Psi_i\,(\sin(\beta\,(x-c_i))-\sinh(\beta\,(x-c_i)))\,\Omega^2\,\rho\,A}{\beta^3}\right) \\
& + \frac{1}{2}\left(\sum_{i=1}^{n_2} \Omega^2\,\Phi_i\,J_i\,(\cosh(\beta\,(x-b_i))+\cos(\beta\,(x-b_i)))\,\mathrm{Heaviside}(x-b_i)\right)
\end{aligned}
$$

$$, \qquad (12) $$

$$
\begin{aligned}
\phi_s(x) = {} & \frac{1}{2}\,\frac{\beta^2\,(-\cosh(\beta\,x)+\cos(\beta\,x))\,Q_s(0)}{A\,\rho\,\Omega^2} - \frac{1}{2}\,\frac{\beta^3\,(\sinh(\beta\,x)+\sin(\beta\,x))\,M_s(0)}{A\,\rho\,\Omega^2} \\
& + \left(\frac{1}{2}\cos(\beta\,x)+\frac{1}{2}\cosh(\beta\,x)\right)\phi_s(0) + \frac{1}{2}\,(\sinh(\beta\,x)-\sin(\beta\,x))\,w_s(0)\,\beta \\
& + \frac{1}{2}\left(\sum_{i=1}^{n_1} \frac{\beta^2\,\mathrm{Heaviside}(x-a_i)\,(\cos(\beta\,(x-a_i))-\cosh(\beta\,(x-a_i)))\,R_i}{\Omega^2\,A\,\rho}\right) \\
& + \frac{1}{2}\left(\sum_{i=1}^{n_2} \frac{\beta^2\,(-\cos(\beta\,(x-b_i))+\cosh(\beta\,(x-b_i)))\,m_i\,\mathrm{Heaviside}(x-b_i)\,\mathrm{W}_i}{A\,\rho}\right) \\
& - \frac{1}{2}\left(\sum_{i=1}^{n_2} \frac{\beta^3\,\Phi_i\,J_i\,(\sin(\beta\,(x-b_i))+\sinh(\beta\,(x-b_i)))\,\mathrm{Heaviside}(x-b_i)}{A\,\rho}\right) \\
& + \left(\sum_{i=1}^{n_3} \frac{(\cosh(\beta\,(x-c_i))+\cos(\beta\,(x-c_i)))\,\mathrm{Heaviside}(x-c_i)\,\Psi_i}{2}\right)
\end{aligned}
$$

$$, \qquad (13) $$

$$w_s(x) = \frac{1}{2}\frac{(\sin(\beta x) - \sinh(\beta x))\,\beta\, Q_s(0)}{\Omega^2\,\rho\, A} + \frac{1}{2}\frac{\beta^2\,(-\cosh(\beta x) + \cos(\beta x))\, M_s(0)}{A\,\rho\,\Omega^2}$$
$$+ \frac{1}{2}\frac{(\sinh(\beta x) + \sin(\beta x))\,\phi_s(0)}{\beta} + \left(\frac{1}{2}\cos(\beta x) + \frac{1}{2}\cosh(\beta x)\right) w_s(0)$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n_1}\frac{\text{Heaviside}(x-a_i)\,(-\sinh(\beta\,(x-a_i)) + \sin(\beta\,(x-a_i)))\,\beta\, R_i}{\Omega^2\,\rho\, A}\right)$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n_2}\frac{(\sinh(\beta\,(x-b_i)) - \sin(\beta\,(x-b_i)))\, m_i\,\text{Heaviside}(x-b_i)\,\beta\, W_i}{\rho\, A}\right)$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n_2}\frac{(-\cosh(\beta\,(x-b_i)) + \cos(\beta\,(x-b_i)))\,\Phi_i\, J_i\,\beta^2\,\text{Heaviside}(x-b_i)}{\rho\, A}\right)$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n_3}\frac{\text{Heaviside}(x-c_i)\,\Psi_i\,(\sin(\beta\,(x-c_i)) + \sinh(\beta\,(x-c_i)))}{\beta}\right) \quad , \qquad (14)$$

where Heaviside(*x*-*a*) is the denotation used in Maple for Heaviside's unit step function moved into the point $x=a$.

Example By using Eqs. (11)-(14), derive the frequency equation of a slender beam of the length *l* on three pin rigid supports where both ends of the beam are supported.

SOLUTION Substituting $n_1=1$, $n_2=0$, $n_3=0$,$a_1=a$, $R_1=R$, $m=\rho.A$ into Eqs. (6)-(9) for this case, we receive:

$$\frac{d}{dx} Q_s(x) = -m\, w_s(x)\,\Omega^2 + R\,\text{Dirac}(x-a) \quad , \qquad (15)$$

$$\frac{d}{dx} M_s(x) = Q_s(x) \quad , \qquad (16)$$

$$\frac{d}{dx}\phi_s(x) = -\frac{M_s(x)}{E\,J} \quad , \qquad (17)$$

$$\frac{d}{dx} w_s(x) = \phi_s(x) \quad , \qquad (18)$$

with boundary conditions:

$$w_s(x)\big|_{x=0} = 0 \quad , \quad M_s(x)\big|_{x=0} = 0 \quad , \qquad (19)$$

$$w_s(x)\big|_{x=l} = 0 \quad , \quad M_s(x)\big|_{x=l} = 0 \quad , \qquad (20)$$

and a deformation condition for the concentrated support at $x = a$ between ends of the beam ($a < l$):

$$w_s(x)\big|_{x=a} = 0 \ . \qquad (21)$$

By substituting Eqs. (12) and (14) into Eqs. (20) and (21), and taking into account Eqs. (19), we obtain a homogenous system of three linear algebraic equations for unknown initial parameters $\phi_s(0)$, $Q_s(0)$ and the amplitude R of the reaction at the concentrated support between ends of the beam, in matrix form:

$$BC = 0 \ , \qquad (22)$$

where

$$B=\begin{bmatrix} \frac{\left(-\frac{1}{2}\sinh(l\,\beta)+\frac{1}{2}\sin(l\,\beta)\right)\Omega^2 m}{\beta^2}, & \frac{1}{2}\sinh(l\,\beta)+\frac{1}{2}\sin(l\,\beta), & -\frac{1}{2}\sinh(\beta\,(-l+a))-\frac{1}{2}\sin(\beta\,(-l+a)) \\ \frac{\frac{1}{2}\sinh(l\,\beta)+\frac{1}{2}\sin(l\,\beta)}{\beta}, & \frac{\left(-\frac{1}{2}\sinh(l\,\beta)+\frac{1}{2}\sin(l\,\beta)\right)\beta}{\Omega^2 m}, & \frac{\left(-\frac{1}{2}\sin(\beta\,(-l+a))+\frac{1}{2}\sinh(\beta\,(-l+a))\right)\beta}{\Omega^2 m} \\ \frac{\frac{1}{2}\sin(\beta\,a)+\frac{1}{2}\sinh(\beta\,a)}{\beta}, & \frac{\left(\frac{1}{2}\sin(\beta\,a)-\frac{1}{2}\sinh(\beta\,a)\right)\beta}{\Omega^2 m}, & 0 \end{bmatrix} \quad (23)$$

and

$$C=\begin{bmatrix}\phi_s(0)\\ Q_s(0)\\ R\end{bmatrix} \quad . \qquad (24)$$

For nontrivial solution to Eq. (22), the matrix (23) must be singular, i.e.:

$$\det(B) = 0 \quad . \qquad (25)$$

By substituting Eq. (23) into Eq. (25) and simplifying, we acquire the frequency equation:

$$-\sinh(l\,\beta)\sin(\beta\,(-l+a))\sin(\beta\,a)+\sin(l\,\beta)\sinh(\beta\,(-l+a))\sinh(\beta\,a) = 0 \quad . \qquad (26)$$

6. Conclusions

Contribution of this paper to modal analysis of slender beams is that the mathematical model for free transverse vibration, i.e. Eqs. (2)-(5), holds true also for the discontinuous shear force, the discontinuous bending moment and the discontinuous slope. Discontinuities in shear force are supposed to be owing to idealized concentrated supports or inertia masses between ends of the beam. Likewise, discontinuities in bending moment are assumed to be due to idealized concentrated moments of inertia situated between ends of the beam. On the contrary, discontinuities in slope are caused by real hinges connecting beam segments. Jump discontinuities in unknown dependently variable quantities have been expressed in corresponding distributional derivatives (2)-(4) where the singular distribution Dirac(x), which is usually denoted as δ(x), is always moved into the point with a discontinuity mentioned, and multiplied by a magnitude of the discontinuity. To be able to find modal shapes of a slender beam analytically with discontinuities mentioned, we have derived Eqs. (6)-(9) for shapes of the shear force, the bending moment, the slope and the deflection. Using the Laplace transform method, we have computed the general solution (11)-(14) containing integration constants in the form of initial parameters.

Computing limits of Eq. (14) at points $x=b_i$, we can express the unknown amplitudes of the deflection W_i as functions of initial parameters. Similarly, computing limits of Eq. (13) at points $x=b_i$ step-by-step, we can express the unknown amplitudes of the slope Φ_i as functions of initial parameters. In order to determine the unknown initial parameters, we must establish four boundary conditions. So as to determine the unknown reactions at concentrated supports between ends of the beam, and amplitudes of discontinuities in the slope at hinges connecting beam segments, we must establish corresponding deformation conditions at these points. These deformation and boundary conditions create all together a homogeneous system of linear equations. The condition of the nontrivial solution to this system is the frequency equation of the beam with discontinuities assumed.

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