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VARIANTS OF NAVIER-STOKES EQUATIONS

F. Pochylý*, S. Fialová, H. Krausová*****

Abstract: *The different variants of the Navier -Stokes equations are presented in the paper. The authors also present new variants to be used for qualitative analysis of the fluid flow. New variants can also be used for the numerical solution, especially for the method of control volumes.*

Keywords: ***Navier-Stokes equations, compressible liquid, bulk viscosity.***

1. Introduction

Nowadays, research in fluid flow is mainly focused on computational simulation. Often, however, a qualitative analysis of the problem is missing. Predicting the behavior of the pressure and the velocity fields can be based on qualitative analysis of the Navier -Stokes equations and continuity equation. Firstly, it is necessary to modify the above mentioned equations to a suitable form for analysis.

Recently, the questions of unsteady fluid motion are being solved. Modified Navier -Stokes equations into a new form might serve for these purposes. This form is suitable for qualitative analysis of the interaction of body with fluids, as well as for the new variant of the method of control volumes.

It is possible that the proposed modifications of the local gravity acceleration forces are already known. In this case, the author believes that the shown diversity of the Navier Stokes equations will serve to the readers.

Used symbols

x_i - Cartesian system coordination, y_i - rotating system coordination, t - time, $\mathbf{u}$ - translation, $\mathbf{v}$ - absolute velocity, $\mathbf{n}$ - outer normal vector to the liquid, $\mathbf{w}$ - relative velocity, $\boldsymbol{\omega}$ - angular velocity, δ_{ij} - Kronecker delta, ε_{ijk} - Levi-Civit tensor, $\boldsymbol{\Omega}$ - $\mathrm{rot}\,\mathbf{v}, \mathrm{rot}\,\mathbf{w}; \Delta\mathbf{v} = -\mathrm{rot}\,\boldsymbol{\Omega} + \mathrm{grad}\,\mathrm{div}\,\mathbf{v}$, η - dynamical viscosity. ξ - bulk viscosity, λ - second viscosity, ρ - density, $\mathbf{H}$ - magnetic field intensity, μ - surroundings permeability, $\mathbf{g}$ - gravity acceleration, $\mathbf{x}$ - (x_1, x_2, x_3), $\mathbf{y}$ - (y_1, y_2, y_3), Y – specific energy, $\mathbf{U}$ - tangential velocity, ν - sound velocity, the sum convention is used in the paper.

2. Compressible liquid

As it is well known that the equation of force equilibrium of macroscopic fluid particle can be written in the form:

$$\rho \frac{dv_i}{dt} - \frac{\partial \sigma_{ij}}{\partial x_j} = G_i \, . \tag{1}$$

* Prof. Ing. František Pochylý, CSc.: Institute of Power Engineering, Brno University of Technology, Technická 2896/2; 616 69, Brno; CZ, e-mail: pochyly@fme.vutbr.cz

** Ing. Simona Fialová, Ph.D.: Institute of Power Engineering, Brno University of Technology; Technická 2896/2; 616 69, *Brno*; CZ, e-mail: fialova@fme.vutbr.cz

*** Ing. Hana Krausová: Institute of Power Engineering, Brno University of Technology; Technická 2896/2; 616 69, Brno; CZ, e-mail: y100881@stud.fme.vutbr.cz

It is solved together with the continuity equation:

$$\frac{dp}{dt} + \rho v^2 \frac{\partial v_i}{\partial x_i} = 0. \qquad (2)$$

It is possible to split the stress tensor σ_{ij} on the bases of superposition (de Groot and Mazur 1962) into two parts:

$$\sigma_{ij} = \tau_{ij} + \Pi_{ij} \qquad (3)$$

The first part τ_{ij} is connected with the so called reversible thermodynamic phenomenon and is represented by the pressure function p, see the term:

$$\tau_{ij} = -p\delta_{ij}. \qquad (4)$$

The irreversible part Π_{ij} includes the damping properties of the liquid. These properties may vary significantly for different types of fluids. Therefore it is necessary to seek constitutive relations for each fluid separately, where the stress tensor Π_{ij} depends on the strain rate tensor. According to this dependence, fluids are divided into so-called Newtonian and the other rheological fluids (e.g. Bingham).

In this paper we consider only the Newtonian fluids that are characterized by dependence:

$$\Pi_{ij} = 2\eta v_{ij} + \lambda\delta_{ij} v_{kk} \qquad (5)$$

where λ is so-called second viscosity (Brdička et al., 2000; de Groot and Mazur, 1962; Pochylý et al., 2011).

Based on the kinetic theory of gases principle, Enskog (Brdička et al., 2000) derived the relationship:

$$\lambda = -\frac{2}{3}\eta. \qquad (6)$$

On the basis of a series of experiments it was found (Pochylý et al. 2011) that this relationship is not endorsed and has to be supplemented by the value of bulk viscosity ξ, so that:

$$\lambda = \xi - \frac{2}{3}\eta. \qquad (7)$$

The results of the experiments suggest that λ strongly depends on the frequency. For water, the following applies:

$$\lambda = \rho\frac{9800}{f}\ ; \quad f < 1000 H_z. \qquad (8)$$

When we expressed $\frac{dv_i}{dt}$ in Euler approach and considering (4), (5), (7), can be (1) written in the form:

$$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial}{\partial x_i}\left[\frac{1}{2}|\mathbf{v}|^2\right] - \rho\varepsilon_{ijk} v_j \Omega_k - (2\eta+\lambda)\frac{\partial^2 v_k}{\partial x_i \partial x_k} + \eta\varepsilon_{ijk}\frac{\partial \Omega_k}{\partial x_j} + \operatorname{grad} p = G_i, \qquad (9)$$

or as a vector:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\operatorname{grad}\left(\frac{1}{2}|\mathbf{v}|^2\right) - \rho\mathbf{v}x\mathbf{\Omega} - (2\eta+\lambda)\operatorname{graddiv}\mathbf{v} + \eta\operatorname{rot}\mathbf{\Omega} + \operatorname{grad} p = \mathbf{G}. \qquad (10)$$

Vector $\mathbf{G}$ on the right side of (10) represents the bulk force. This effect is caused by external gravitational fields $\mathbf{G}_g$, or electromagnetic fields $\mathbf{G}_M$. For these forces the relations hold:

$$\mathbf{G}_g = \rho \mathbf{g}; \qquad \mathbf{G}_M = \mu \operatorname{rot} \mathbf{H} \times \mathbf{H} \tag{11}$$

In the text that follows, we neglect $\mathbf{G}_M$, as the influence of magnetic fields deserves a special attention.

In many technical applications, the specific energy Y is of an importance, defined by term:

$$Y = \frac{|\mathbf{v}|^2}{2} + \frac{p}{\rho} - \mathbf{g} \cdot \mathbf{x} . \tag{12}$$

For these purposes, it is appropriate to modify the expression for the gravitational force $\mathbf{G}_g$. We propose a new shape for $\mathbf{G}_g$. It is easy to show that the following holds:

$$\mathbf{G}_g = \rho \operatorname{grad}(\mathbf{g} \cdot \mathbf{x}). \tag{13}$$

By this definition of gravitational forces, the equation (10) can be modified by using the specific energy:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \operatorname{grad} Y - \rho \mathbf{v} \times \mathbf{\Omega} - (2\eta + \lambda) \operatorname{graddiv} \mathbf{v} + \eta \operatorname{rot} \mathbf{\Omega} = 0 . \tag{14}$$

When solving small oscillations of a body in the compressible fluid, equations (14) could be written using the translation $\mathbf{u}$. If we decompose the liquid field into its stationary and non-stationary components induced by small oscillations of the body, it can be written that (Pochylý, 2009).

$$p = -\rho v^2 \operatorname{div} \mathbf{u} \tag{15}$$

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 + \mathbf{c}; \quad \mathbf{v}_0 = \mathbf{v}_0(\mathbf{x}); \quad \mathbf{c} = \mathbf{c}(\mathbf{x}, t) \\ \mathbf{\Omega} &= \mathbf{\Omega}_0 + \boldsymbol{\omega}; \quad \mathbf{\Omega}_0 = \mathbf{\Omega}_0(\mathbf{x}); \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x}, t) \end{aligned} . \tag{16}$$

If we neglect in the equation (14) the small velocity components and when we express it using the translation

$$\mathbf{c} = \frac{\partial \mathbf{u}}{\partial t}, \tag{17}$$

it (1.10) can be written in the form:

$$\rho \frac{\partial \mathbf{c}}{\partial t} + \rho \operatorname{grad}(\mathbf{v}_0 \mathbf{c}) - \rho \mathbf{v}_0 \times \boldsymbol{\omega} - \rho \mathbf{c} \times \mathbf{\Omega}_0 - (2\eta + \lambda) \operatorname{graddiv} \mathbf{c} + \eta \operatorname{rot} \boldsymbol{\omega} - \rho v^2 \operatorname{graddiv} \mathbf{u} = 0 \tag{18}$$

This form of the equation allows using the ANSYS software environment, which is commonly used in the classical mechanics problems solutions.

3. Incompressible liquid

The incompressibility condition is expressed by the continuity equation in the form:

$$\operatorname{div} \mathbf{v} = 0 . \tag{19}$$

on the basis of this equation, (14) can be simplified to the form:

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho\,\mathrm{grad}\,Y-\rho\mathbf{v}\times\mathbf{\Omega}+\eta\,\mathrm{rot}\,\mathbf{\Omega}=0.. \tag{20}$$

Equation (20) is easily written in the index symbolism in two versions:

$$\rho\frac{\partial v_i}{\partial t}+\rho\frac{\partial Y}{\partial x_i}-\rho\varepsilon_{ijk}\varepsilon_{klm}v_j\frac{\partial v_m}{\partial x_j}-\eta\frac{\partial^2 v_i}{\partial x_j\partial x_j}=0. \tag{21}$$

or in a simpler form:

$$\frac{\partial v_i}{\partial t}+\frac{\partial}{\partial x_j}\left(v_iv_j+\delta_{ij}\frac{p}{\rho}-\delta_{ij}g_kx_k-\frac{\eta}{\rho}\frac{\partial v_i}{\partial x_j}\right)=0. \tag{22}$$

For the investigation of the force effects between the liquid and the body it is suitable to write the equation (22) in the basic form (1), taking into account the substance's derivative.

$$\frac{dv_i}{dt}=\frac{\partial v_i}{\partial t}+\frac{\partial v_i}{\partial x_j}v_j \tag{23}$$

Under these assumptions, (1) can be written for example in the form:

$$\rho\frac{\partial v_i}{\partial t}+\rho\frac{\partial}{\partial x_j}\left(v_iv_j-\delta_{ij}g_kx_k\right)-\frac{\partial\sigma_{ij}}{\partial x_j}=0. \tag{24}$$

Assuming that the body is made up of multiple contiguous areas bounded by the surface S and the liquid is sealed in the field V in between Γ and S, you can write for the force $\mathbf{F}$ the relationship

$$(\mathbf{F})_i=F_i=-\int_S\sigma_{ij}n_j\,dS. \tag{25}$$

After integration of (24) using the Gauss-Ostrogradsky theorem we obtain:

$$F_i=-\rho\int_V\frac{\partial v_i}{\partial t}dV-\rho\int_{S\cup\Gamma}\left(\rho v_iv_j-\delta_{ij}g_kx_k\right)n_j\,d\Theta+\int_\Gamma\sigma_{ij}n_j\,d\Gamma\,. \tag{26}$$

The forces analysis in the unsteady motion of the body in the liquid according to the relation (26) is complex, given that you cannot reliably determine the effect of volume integrals.

This deficiency can be removed by following the proposed modification. Easy to see that, assuming (19), it holds:

$$\frac{\partial v_i}{\partial t}=\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial t}x_i\right) \tag{27}$$

Using this adjustment, (24) as well as (26) can be significantly simplified. Substituting into (24), (26) we obtain:

$$\frac{\partial}{\partial x_j}\left[\rho\frac{\partial v_j}{\partial t}x_i+\rho v_iv_j-\sigma_{ij}-\delta_{ij}\rho g_kx_k\right]=0 \tag{28}$$

$$F_i = -\int_{S \cup \Gamma} \rho \left(\frac{\partial v_j}{\partial t} x_i + v_i v_j - \delta_{ij} g_k x_k \right) n_j d\Theta + \int_{\Gamma} \sigma_{ij} n_j d\Gamma. \tag{29}$$

The right side of the expression (29) is no longer dependent on the volume integrals and determines the additional effects of the liquid on the body.

Equation (28) can serve as a starting point for the solution of the non-stationary fluid mechanics problems, including unsteady body motion in the liquid.

For example, for a method of control volumes: By the integrating of (28) over the control volume ΔV bounded by the surface ΔS , the following equation can be obtained:

$$\int_{\Delta S} \left(\rho \frac{\partial v_j}{\partial t} x_i + \rho v_i v_j - \sigma_{ij} \delta_{ij} \rho g_k x_k \right) n_j d\Delta S = 0 \tag{30}$$

4. Relative space

Suppose that the coordinate system (y_i) rotates to x_i by angular velocity $\boldsymbol{\omega}(\omega,0,0)$. Navier -Stokes equations in the system (y_i) can be written for incompressible liquid for example in the following two options:

$$\rho \frac{\partial \mathbf{w}}{\partial t} + \rho \boldsymbol{\omega} x(\boldsymbol{\omega} \times \mathbf{y}) - 2\rho \boldsymbol{\omega} \times \mathbf{w} - \eta \Delta \mathbf{w} + \text{grad}(p - \rho \mathbf{g} \mathbf{y}) = 0 \tag{31}$$

$$\rho \frac{\partial \mathbf{w}}{\partial t} + \rho \, \text{grad}[Y - U(\mathbf{v} \cdot \mathbf{U}) - \mathbf{g} \cdot \mathbf{y}] - \mathbf{w} x(2\boldsymbol{\omega} + \text{rot}\, \mathbf{w}) - \eta \Delta \mathbf{w} = 0 \tag{32}$$

Also in the component form:

$$\rho \frac{\partial w_i}{\partial t} + \rho \frac{\partial}{\partial y_j}(w_i w_j) + \rho \omega^2 \varepsilon_{i1k} \varepsilon_{k1m} y_m + 2\rho \omega \varepsilon_{i1k} w_k - \frac{\partial^2 w_i}{\partial y_j \partial y_j} + \frac{\partial}{\partial y_i}(p - \rho g_j y_j) = 0 \tag{33}$$

$$\rho \frac{\partial w_i}{\partial t} + \rho \frac{\partial}{\partial y_j} \left[w_i w_j - \frac{1}{2} |\mathbf{u}|^2 \delta_{ij} + 2u_i w_j - \frac{\eta}{\rho} \frac{\partial w_i}{\partial y_j} + \delta_{ij} \left(\frac{p}{\rho} - g_k y_k \right) \right] = 0 \tag{34}$$

Adjusting

$$\frac{\partial w_i}{\partial t} = \frac{\partial}{\partial y_j} \left(\frac{\partial w_j}{\partial t} y_i \right) \tag{35}$$

Can be (34) rewritten into simpler form:

$$\frac{\partial}{\partial y_j} \left[\frac{\partial w_j}{\partial t} y_i + w_i w_j - \frac{1}{2} |\mathbf{u}|^2 \delta_{ij} + 2u_i w_j - \frac{\eta}{\rho} \frac{\partial w_i}{\partial y_j} + \delta_{ij} \left(\frac{p}{\rho} - g_k y_k \right) \right] = 0 \tag{36}$$

The expression (36) expresses a new notation of Navier -Stokes equations in a rotating coordinate system suitable for further analysis

5. Conclusions

The paper presents some new findings, which can be used both for qualitative analysis of the flow, as well as to simplify the method of control volumes. The volume or the second viscosity is of particular importance. They can get more accurate results for reliable modeling of unsteady pressure pulsations. The essential findings presented in this study can be formulated as follows:

$$\lambda = \rho \frac{9800}{f}; \quad f < 1000 H_z \quad \text{for water}$$

$$\frac{\partial y_i}{\partial t} = \frac{\partial}{\partial x_j} \left(\frac{\partial y_i}{\partial t} x_i \right); \qquad g_i = \frac{\partial}{\partial x_i} (\mathbf{g} \cdot \mathbf{x})$$

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