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LTB RESISTANCE OF BEAMS INFLUENCED BY PLASTIC RESERVE OR LOCAL BUCKLING

Y. Koleková*, I. Baláž**

Abstract: *The results of original procedure concerning calculation of the values of the critical moments M_{cr} using approximate formulae convenient for educational and standardization purposes are presented. The authors formulae are today used in many international and national standardization documents (e.g. in several Eurocodes and their National Annexes), in engineering practice of 32 countries and in educational process at many Universities (e.g. in EPU de São Paulo, EPFL Lausanne, etc.). New C_1 values are presented for beams under combination of uniform loading and unequal end moments. Development of useful modification of Eurocode formulae for lateral torsional buckling resistance $M_{b,Rd}$, which enable to show very clearly the influence of plastic resistance and local buckling on LTB resistance of metal (steel, stainless steel, aluminium) and timber beams calculated by any of 4 Eurocode methods.*

Keywords: ***Critical moment, LTB resistance, steel and aluminium beams, Eurocodes.***

1. Introduction

In different Eurocodes: EN 1993-1-1 (2005), EN 1993-1-4 (2006) for steel, EN 1995-1-1 (2004) for timber, EN 1994-1-1 (2004) for composite steel and concrete and EN 1999-1-1 (2007) for aluminium alloys structures, different ways of calculations of critical moments and resistances of laterally unrestrained beams are used. It was shown in the papers Baláž, I. – Koleková, Y. (2000 b, c, 2002 a, b, 2004 a, b) that rules of different Eurocodes concerning lateral torsional buckling could be unified. In this paper EN 1993-1-1 (2005) (ENV 1993-1-1, 1992), EN 1993-1-4 (2006) and EN 1999-1-1 (2007) will be analysed. Lateral torsional buckling of timber structures including analysis of EN 1995-1-1 rules was analysed in Baláž, I. – Koleková, Y. (2004 a, b) and in Baláž (2005).

2. Critical moment M_{cr}

Critical moment M_{cr} is an important quantity, which is needed for calculation of relative slenderness $\bar{\lambda}_{LT}$. The value of M_{cr} may be calculated: (i) more exactly by using a computer program, or (ii) approximately by using various less or more exact approximate formulae of different authors or standards, which have different forms.

The approximate Clark-Mrázik 3-factors formula (see Baláž, I. – Koleková, Y., 2000 c, 2002 a, b) has the best form and it is much more convenient than 1-factor formulae, see e.g. in Roik, K. – Carl, J. – Lindner, J. (1972). Similar 3-factors formula was used also in European prestandards ENV 1993-1-1 (1992) and in ENV 1999-1-1 (1998). Authors showed several times (Baláž, I., 1999, Baláž, I. – Koleková, Y., 1999, 2000, 2002, Koleková, Y., 1999) that using of values of factors C_1, C_2, C_3 taken from tables of ENV 1993-1-1 (1992) or from ENV 1999-1-1 (1998) leads in many cases to incorrect values of critical moments. Despite of this fact the factors C_1, C_2, C_3 defined in ENV 1993-1-1 (1992) and ENV 1999-1-1 (1998) are still used in practise, in many good books (e.g. in Hirt, M.A. – Bez, R., 1998, Hirt, M.A. – Bez, R. – Nussbaumer, A., 2007)) and also in Access Steel available in Internet. Authors criticized their use in drafts prEN 1993-1-1 and prEN 1999-1-1 and consequently the informative annex containing the ENV tables was completely removed from EN 1993-1-1 (2005),

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which now does not contain any M_{cr} formulae. New more general formulae and tables enabling to compute M_{cr} developed by authors in Baláž, I. – Koleková, Y. (2000 b) were fully accepted for Annex I of drafts and later also of final version of EN 1999-1-1 (2007). In Fruchtengarten, J. (2005) a lot of various formulae were evaluated in the frame of parametric study by comparing them with exact results of program PEFSYS and it was concluded that proposal of Baláž, I. – Koleková, Y. (2000 b) gives the most exact results. Factors C_1, C_2, C_3 computed by authors Baláž, I. – Koleková, Y. (2000 b) and their formula for calculating of elastic critical moment M_{cr} are used in several National Annexes, e.g. in Slovak (STN EN 1993-1-1/NA, 2007), Czech (ČSN EN 1993-1-1, 2006), Austrian (ÖNORM B 1993-1-1, 2007) and Belgian (2005) National Annexes to Eurocode EN 1993-1-1 (2005), because this Eurocode gives no details of calculation of M_{cr}. Authors results are used also in tables used in the following publications: Deutscher Ausschluß für Stahlbau (2005), ECCS Technical Committee 8 – Stability (2006), Design Manual For Structural Stainless Steel. (2006), Excerpt from the Background Document to EN 1993-1-1 (2010).

According to authors proposal the elastic critical moment M_{cr} can be computed from the formula

$$M_{\mathrm{cr}} = \mu_{\mathrm{cr}} \frac{\pi\sqrt{EI_{\mathrm{z}}GI_{\mathrm{t}}}}{L} \tag{1}$$

where
$$\mu_{\mathrm{cr}} = \frac{C_1}{k_{\mathrm{z}}}\left[\sqrt{1+\kappa_{\mathrm{wt}}^2+(C_2\zeta_{\mathrm{g}}-C_3\zeta_{\mathrm{j}})^2}-(C_2\zeta_{\mathrm{g}}-C_3\zeta_{\mathrm{j}})\right] \tag{2}$$

three non-dimensional parameters are

$$\kappa_{\mathrm{wt}} = \frac{\pi}{k_{\mathrm{w}}L}\sqrt{\frac{EI_{\mathrm{w}}}{GI_{\mathrm{t}}}}, \quad \zeta_{\mathrm{g}} = \frac{\pi z_{\mathrm{g}}}{k_{\mathrm{z}}L}\sqrt{\frac{EI_{\mathrm{z}}}{GI_{\mathrm{t}}}}, \quad \zeta_{\mathrm{j}} = \frac{\pi z_{\mathrm{j}}}{k_{\mathrm{z}}L}\sqrt{\frac{EI_{\mathrm{z}}}{GI_{\mathrm{t}}}} \tag{3}$$

and three factors C_1, C_2, C_3 depend on the loadings, end restraint conditions, shape of the cross-section and C_1 also on torsional properties. The details and numerical values see in Tables 1-4 or in Tables 1 and 2 in Baláž, I. – Koleková, Y. (2000 b), Tables I.1 - I.4 in EN 1999-1-1 (2007) or Tables NB.3.1-NB.3.4 in STN EN 1993-1-1/NA (2007).

Our general formula (1) becomes very approximate formula used in German standard DIN 18 800 (1990, 2008) in the case of double symmetric cross-section ($z_{\mathrm{j}} = 0$ mm) when our more refined values of factor C_2 are replaced by rough value 0,5 used in DIN 18 800 (1990, 2008). The meaning of our factor C_1 is the same as the meaning of the factor ζ used in DIN 18 800 (1990, 2008), Table 10.

Factor C_1 depends on bending moment distribution, boundary conditions and parameter κ_{wt}. In Baláž, I. – Koleková, Y. (2000 b), in EN 1999-1-1 (2007) or in STN EN 1993-1-1/NA (2007) a linear interpolation between values $C_{1,0} = C_1(\kappa_{\mathrm{wt}} = 0)$ and $C_{1,1} = C_1(\kappa_{\mathrm{wt}} = 1)$ is proposed. For $\kappa_{\mathrm{wt}} \geq 1$ it is proposed in Baláž, I. – Koleková, Y. (2000 b), in EN 1999-1-1 (2007) and in STN EN 1993-1-1/NA (2007) to use an approximation $C_1(\kappa_{\mathrm{wt}} \geq 1) \approx C_{1,1} = C_1(\kappa_{\mathrm{wt}} = 1)$. For many loading cases the difference between values $C_{1,0} = C_1(\kappa_{\mathrm{wt}} = 0)$ and $C_{1,1} = C_1(\kappa_{\mathrm{wt}} = 1)$ is negligible, that is why many authors of various publications even do not inform, which value of C_1 they use. Eurocode EN 1993-1-1 (2005) uses C_1 in Table A.1, formulae BB.5 and BB.9 without any definition . Here it is an advise to users of EN 1993-1-1 (2005): you can use relevant values of C_1 given in Baláž, I. – Koleková, Y. (2000 b), in EN 1999-1-1 (2007) or in STN EN 1993-1-1/NA (2007), or you can use an approximation $C_1 = (k_c)^{-2}$, where k_c is a correction factor for relevant moment distribution (see Table 6.6 in EN 1993-1-1 (2005)).

Tab. 1: Values of factors C_1 and C_3 corresponding to various end moment ratios ψ, values of buckling length factor k_z and cross-section parameters ψ_f and κ_{wt}.

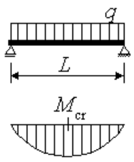
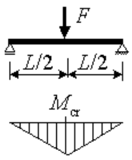
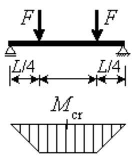
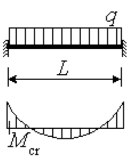
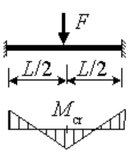
End moment loading of the simply supported beam with buckling length factors $k_y = 1$ for major axis bending and $k_w = 1$ for torsion

<table>
<tr><th rowspan="3">Loading and support conditions.
Cross-section monosymmetry factor ψ_f</th><th rowspan="3">Bending moment diagram.
End moment ratio ψ.
M-side ψM-side</th><th rowspan="3">k_z 2)</th><th colspan="6">Values of factors</th></tr>
<tr><th colspan="2">C_1 1)</th><th colspan="4">C_3</th></tr>
<tr><th>$C_{1,0}$</th><th>$C_{1,1}$</th><th>$\psi_f = -1$</th><th>$-0,9 \le \psi_f \le 0$</th><th>$0 \le \psi_f \le 0,9$</th><th>$\psi_f = 1$</th></tr>
<tr><td rowspan="16">M ψM
L
$k_y = 1$, $k_w = 1$
Beam M-side:
$\psi_f \ge 0$
$\psi_f \le 0$</td><td rowspan="4">M_{cr} $\psi = +1$</td><td>1,0</td><td>1,000</td><td>1,000</td><td colspan="4">1,000</td></tr>
<tr><td>0,7L</td><td>1,016</td><td>1,100</td><td colspan="2">1,025</td><td colspan="2">1,000</td></tr>
<tr><td>0,7R</td><td>1,016</td><td>1,100</td><td colspan="2">1,025</td><td colspan="2">1,000</td></tr>
<tr><td>0,5</td><td>1,000</td><td>1,127</td><td colspan="4">1,019</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = +3/4$</td><td>1,0</td><td>1,139</td><td>1,141</td><td colspan="4">1,000</td></tr>
<tr><td>0,7L</td><td>1,210</td><td>1,313</td><td colspan="2">1,050</td><td colspan="2">1,000</td></tr>
<tr><td>0,7R</td><td>1,109</td><td>1,201</td><td colspan="4">1,000</td></tr>
<tr><td>0,5</td><td>1,139</td><td>1,285</td><td colspan="4">1,017</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = +1/2$</td><td>1,0</td><td>1,312</td><td>1,320</td><td>1,150</td><td colspan="3">1,000</td></tr>
<tr><td>0,7L</td><td>1,480</td><td>1,616</td><td colspan="2">1,160</td><td colspan="2">1,000</td></tr>
<tr><td>0,7R</td><td>1,213</td><td>1,317</td><td colspan="4">1,000</td></tr>
<tr><td>0,5</td><td>1,310</td><td>1,482</td><td>1,150</td><td colspan="3">1,000</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = +1/4$</td><td>1,0</td><td>1,522</td><td>1,551</td><td>1,290</td><td colspan="3">1,000</td></tr>
<tr><td>0,7L</td><td>1,853</td><td>2,059</td><td>1,600</td><td>1,260</td><td colspan="2">1,000</td></tr>
<tr><td>0,7R</td><td>1,329</td><td>1,467</td><td colspan="4">1,000</td></tr>
<tr><td>0,5</td><td>1,516</td><td>1,730</td><td>1,350</td><td colspan="3">1,000</td></tr>
<tr><td rowspan="20">M ψM
L
$k_y = 1$, $k_w = 1$
Beam M-side:
$\psi_f \le 0$
$\psi_f \ge 0$
$\psi_f = \frac{I_{fc} - I_{ft}}{I_{fc} + I_{ft}}$</td><td rowspan="4">$M_{cr}$ $\psi = 0$</td><td>1,0</td><td>1,770</td><td>1,847</td><td>1,470</td><td colspan="3">1,000</td></tr>
<tr><td>0,7L</td><td>2,331</td><td>2,683</td><td>2,000</td><td>1,420</td><td colspan="2">1,000</td></tr>
<tr><td>0,7R</td><td>1,453</td><td>1,592</td><td colspan="4">1,000</td></tr>
<tr><td>0,5</td><td>1,753</td><td>2,027</td><td>1,500</td><td colspan="3">1,000</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = -1/4$</td><td>1,0</td><td>2,047</td><td>2,207</td><td>1,65</td><td>1,000</td><td colspan="2">0,850</td></tr>
<tr><td>0,7L</td><td>2,827</td><td>3,322</td><td>2,40</td><td>1,550</td><td>0,850</td><td>-0,30</td></tr>
<tr><td>0,7R</td><td>1,582</td><td>1,748</td><td>1,38</td><td>0,850</td><td>0,700</td><td>0,20</td></tr>
<tr><td>0,5</td><td>2,004</td><td>2,341</td><td>1,75</td><td>1,000</td><td>0,650</td><td>-0,25</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = -1/2$</td><td>1,0</td><td>2,331</td><td>2,591</td><td>1,85</td><td>1,000</td><td>$1,3 - 1,2\psi_f$</td><td>-0,70</td></tr>
<tr><td>0,7L</td><td>3,078</td><td>3,399</td><td>2,70</td><td>1,450</td><td>$1 - 1,2\psi_f$</td><td>-1,15</td></tr>
<tr><td>0,7R</td><td>1,711</td><td>1,897</td><td>1,45</td><td>0,780</td><td>$0,9 - 0,75\psi_f$</td><td>-0,53</td></tr>
<tr><td>0,5</td><td>2,230</td><td>2,579</td><td>2,00</td><td>0,950</td><td>$0,75 - \psi_f$</td><td>-0,85</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = -3/4$</td><td>1,0</td><td>2,547</td><td>2,852</td><td>2,00</td><td>1,000</td><td>$0,55 - \psi_f$</td><td>-1,45</td></tr>
<tr><td>0,7L</td><td>2,592</td><td>2,770</td><td>2,00</td><td>0,850</td><td>$0,23 - 0,9\psi_f$</td><td>-1,55</td></tr>
<tr><td>0,7R</td><td>1,829</td><td>2,027</td><td>1,55</td><td>0,700</td><td>$0,68 - \psi_f$</td><td>-1,07</td></tr>
<tr><td>0,5</td><td>2,352</td><td>2,606</td><td>2,00</td><td>0,850</td><td>$0,35 - \psi_f$</td><td>-1,45</td></tr>
<tr><td rowspan="4">M_{cr} $\psi = -1$</td><td>1,0</td><td>2,555</td><td>2,733</td><td>2,00</td><td colspan="2">$-\psi_f$</td><td>-2,00</td></tr>
<tr><td>0,7L</td><td>1,921</td><td>2,103</td><td>1,55</td><td>0,380</td><td>-0,580</td><td>-1,55</td></tr>
<tr><td>0,7R</td><td>1,921</td><td>2,103</td><td>1,55</td><td>0,580</td><td>-0,380</td><td>-1,55</td></tr>
<tr><td>0,5</td><td>2,223</td><td>2,390</td><td>1,88</td><td>$0,125 - 0,7\psi_f$</td><td>$-0,125 - 0,7\psi_f$</td><td>-1,88</td></tr>
</table>

1) $C_1 = C_{1,0} + (C_{1,1} - C_{1,0})\kappa_{wt} \le C_{1,1}$, ($C_1 = C_{1,0}$ for $\kappa_{wt} = 0$, $C_1 = C_{1,1}$ for $\kappa_{wt} \ge 1$)

2) $0,7L$ = left end fixed, $0,7R$ = right end fixed

Tab. 2: Values of factors C_1, C_2 and C_3 corresponding to various transverse loading cases, values of buckling length factors k_y, k_z, k_w, cross-section monosymmetry factor ψ_f and torsion parameter κ_{wt}.

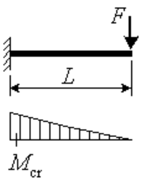
Loading and support conditions	Buckling length factors			Values of factors							
	k_y	k_z	k_w	C_1 [1)]		C_2			C_3		
				$C_{1,0}$	$C_{1,1}$	⊥ $\psi_f=-1$	I I I $-0,9 \le \psi_f \le 0,9$	T $\psi_f=1$	⊥ $\psi_f=-1$	I I I $-0,9 \le \psi_f \le 0,9$	T $\psi_f=1$
	1	1	1	1,127	1,132	0,33	0,459	0,50	0,93	0,525	0,38
	1	1	0,5	1,128	1,231	0,33	0,391	0,50	0,93	0,806	0,38
	1	0,5	1	0,947	0,997	0,25	0,407	0,40	0,84	0,478	0,44
	1	0,5	0,5	0,947	0,970	0,25	0,310	0,40	0,84	0,674	0,44
	1	1	1	1,348	1,363	0,52	0,553	0,42	1,00	0,411	0,31
	1	1	0,5	1,349	1,452	0,52	0,580	0,42	1,00	0,666	0,31
	1	0,5	1	1,030	1,087	0,40	0,449	0,42	0,80	0,338	0,31
	1	0,5	0,5	1,031	1,067	0,40	0,437	0,42	0,80	0,516	0,31
	1	1	1	1,038	1,040	0,33	0,431	0,39	0,93	0,562	0,39
	1	1	0,5	1,039	1,148	0,33	0,292	0,39	0,93	0,878	0,39
	1	0,5	1	0,922	0,960	0,28	0,404	0,30	0,88	0,539	0,50
	1	0,5	0,5	0,922	0,945	0,28	0,237	0,30	0,88	0,772	0,50
						$\psi_f=-1$	$-0,5 \le \psi_f \le 0,5$	$\psi_f=1$	$\psi_f=-1$	$-0,5 \le \psi_f \le 0,5$	$\psi_f=1$
	0,5	1	1	2,576	2,608	1,00	1,562	0,15	1,00	-0,859	-1,99
	0,5	0,5	1	1,490	1,515	0,56	0,900	0,08	0,61	-0,516	-1,20
	0,5	0,5	0,5	1,494	1,746	0,56	0,825	0,08	0,61	0,002712	-1,20
	0,5	1	1	1,683	1,726	1,20	1,388	0,07	1,15	-0,716	-1,35
	0,5	0,5	1	0,936	0,955	0,69	0,763	0,03	0,64	-0,406	-0,76
	0,5	0,5	0,5	0,937	1,057	0,69	0,843	0,03	0,64	-0,0679	-0,76

1) $C_1 = C_{1,0} + (C_{1,1} - C_{1,0})\kappa_{wt} \le C_{1,1}$, ($C_1 = C_{1,0}$ for $\kappa_{wt} = 0$, $C_1 = C_{1,1}$ for $\kappa_{wt} \ge 1$).

2) Parameter ψ_f refers to the middle of the span.

3) Values of critical moments M_{cr} refer to the cross section, where M_{max} is located

Tab. 3: Relative non-dimensional critical moment μ_{cr} for cantilever $(k_y = k_z = k_w = 2)$ loaded by concentrated tip load F.

Loading and support conditions	$\frac{\pi}{L}\sqrt{\frac{EI_w}{GI_t}} = k_w\kappa_{wt} = \kappa_{wt0}$	$\frac{\pi z_g}{L}\sqrt{\frac{EI_z}{GI_t}} = k_z\zeta_g = \zeta_g$	(T) (C) (C) (T)		$\frac{\pi z_j}{L}\sqrt{\frac{EI_z}{GI_t}} = k_z\zeta_j = \zeta_{j0}$			(C) (T) (T) (C)	
			-4	-2	-1	0	1	2	4
	0	4	0,107	0,156	0,194	0,245	0,316	0,416	0,759
		2	0,123	0,211	0,302	0,463	0,759	1,312	4,024
		0	0,128	0,254	0,478	1,280	3,178	5,590	10,730
		-2	0,129	0,258	0,508	1,619	3,894	6,500	11,860
		-4	0,129	0,258	0,511	1,686	4,055	6,740	12,240
	0,5	4	0,151	0,202	0,240	0,293	0,367	0,475	0,899
		2	0,195	0,297	0,393	0,560	0,876	1,528	5,360
		0	0,261	0,495	0,844	1,815	3,766	6,170	11,295
		-2	0,329	0,674	1,174	2,423	4,642	7,235	12,595
		-4	0,364	0,723	1,235	2,529	4,843	7,540	13,100
	1	4	0,198	0,257	0,301	0,360	0,445	0,573	1,123
		2	0,268	0,391	0,502	0,691	1,052	1,838	6,345
		0	0,401	0,750	1,243	2,431	4,456	6,840	11,920
		-2	0,629	1,326	2,115	3,529	5,635	8,115	13,365
		-4	0,777	1,474	2,264	3,719	5,915	8,505	13,960
	2	4	0,335	0,428	0,496	0,588	0,719	0,916	1,795
		2	0,461	0,657	0,829	1,111	1,630	2,698	7,815
		0	0,725	1,321	2,079	3,611	5,845	8,270	13,285
		-2	1,398	3,003	4,258	5,865	7,845	10,100	15,040
		-4	2,119	3,584	4,760	6,360	8,385	10,715	15,825
	4	4	0,845	1,069	1,230	1,443	1,739	2,168	3,866
		2	1,159	1,614	1,992	2,569	3,498	5,035	10,345
		0	1,801	3,019	4,231	6,100	8,495	11,060	16,165
		-2	3,375	6,225	8,035	9,950	11,975	14,110	18,680
		-4	5,530	8,130	9,660	11,375	13,285	15,365	19,925

a) For $z_j = 0$, $z_g = 0$ and $\kappa_{wt0} \le 8$: $\mu_{cr} = 1{,}27 + 1{,}14\,\kappa_{wt0} + 0{,}017\,\kappa_{wt0}^2$.

b) For $z_j = 0$, $-4 \le \zeta_g \le 4$ and $\kappa_{wt} \le 4$, μ_{cr} may be calculated also from formulae (I.7) and (I.8), where the following approximate values of the factors C_1, C_2 should be used for the cantilever under tip load F:

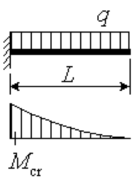
$$C_1 = 2{,}56 + 4{,}675\,\kappa_{wt} - 2{,}62\,\kappa_{wt}^2 + 0{,}5\,\kappa_{wt}^3, \text{ if } \kappa_{wt} \le 2$$

$$C_1 = 5{,}55 \text{ if } \kappa_{wt} > 2$$

$$C_2 = 1{,}255 + 1{,}566\,\kappa_{wt} - 0{,}931\,\kappa_{wt}^2 + 0{,}245\,\kappa_{wt}^3 - 0{,}024\,\kappa_{wt}^4, \text{ if } \zeta_g \ge 0$$

$$C_2 = 0{,}192 + 0{,}585\,\kappa_{wt} - 0{,}054\,\kappa_{wt}^2 - (0{,}032 + 0{,}102\,\kappa_{wt} - 0{,}013\,\kappa_{wt}^2)\,\zeta_g, \text{ if } \zeta_g < 0$$

Tab. 4: Relative non-dimensional critical moment μ_{cr} for cantilever $(k_{\mathrm{y}} = k_{\mathrm{z}} = k_{\mathrm{w}} = 2)$ loaded by uniformly distributed load q

Loading and support conditions	$\frac{\pi}{L}\sqrt{\frac{EI_{\mathrm{w}}}{GI_{\mathrm{t}}}} = k_w \kappa_{wt} = \kappa_{wt0}$	$\frac{\pi z_{\mathrm{g}}}{L}\sqrt{\frac{EI_{\mathrm{z}}}{GI_{\mathrm{t}}}} = k_z \zeta_g = \zeta_{g0}$	(T) (C)	(C) (T)	$\frac{\pi z_{\mathrm{j}}}{L}\sqrt{\frac{EI_{\mathrm{z}}}{GI_{\mathrm{t}}}} = k_z \zeta_j = \zeta_{j0}$			(C) (T)	(T) (C)
			-4	**-2**	**-1**	**0**	**1**	**2**	**4**
	0	4	0,113	0,173	0,225	0,304	0,431	0,643	1,718
		2	0,126	0,225	0,340	0,583	1,165	2,718	13,270
		0	0,132	0,263	0,516	2,054	6,945	12,925	25,320
		-2	0,134	0,268	0,537	3,463	10,490	17,260	30,365
		-4	0,134	0,270	0,541	4,273	12,715	20,135	34,005
	0,5	4	0,213	0,290	0,352	0,443	0,586	0,823	2,046
		2	0,273	0,421	0,570	0,854	1,505	3,229	14,365
		0	0,371	0,718	1,287	3,332	8,210	14,125	26,440
		-2	0,518	1,217	2,418	6,010	12,165	18,685	31,610
		-4	0,654	1,494	2,950	7,460	14,570	21,675	35,320
	1	4	0,336	0,441	0,522	0,636	0,806	1,080	2,483
		2	0,449	0,663	0,865	1,224	1,977	3,873	15,575
		0	0,664	1,263	2,172	4,762	9,715	15,530	27,735
		-2	1,109	2,731	4,810	8,695	14,250	20,425	33,075
		-4	1,623	3,558	6,025	10,635	16,880	23,555	36,875
	2	4	0,646	0,829	0,965	1,152	1,421	1,839	3,865
		2	0,885	1,268	1,611	2,185	3,282	5,700	18,040
		0	1,383	2,550	4,103	7,505	12,770	18,570	30,570
		-2	2,724	6,460	9,620	13,735	18,755	24,365	36,365
		-4	4,678	8,635	11,960	16,445	21,880	27,850	40,400
	4	4	1,710	2,168	2,500	2,944	3,565	4,478	8,260
		2	2,344	3,279	4,066	5,285	7,295	10,745	23,150
		0	3,651	6,210	8,845	13,070	18,630	24,625	36,645
		-2	7,010	13,555	17,850	22,460	27,375	32,575	43,690
		-4	12,270	18,705	22,590	26,980	31,840	37,090	48,390

a) For $z_{\mathrm{j}} = 0$, $z_g = 0$ and $\kappa_{wt0} \le 8$: $\mu_{cr} = 2{,}04 + 2{,}68\,\kappa_{wt0} + 0{,}021\,\kappa_{wt0}^2$.

b) For $z_{\mathrm{j}} = 0$, $-4 \le \zeta_{\mathrm{g}} \le 4$ and $\kappa_{\mathrm{wt}} \le 4$, μ_{cr} may be calculated also from formula (I.7) and (I.8),

where the following approximate values of the factors C_1, C_2 should be used for the cantilever under uniform load q:

$$C_1 = 4{,}11 + 11{,}2\,\kappa_{\mathrm{wt}} - 5{,}65\,\kappa_{\mathrm{wt}}^2 + 0{,}975\,\kappa_{\mathrm{wt}}^3, \text{ if } \kappa_{\mathrm{wt}} \le 2$$

$$C_1 = 12 \text{ if } \kappa_{\mathrm{wt}} > 2$$

$$C_2 = 1{,}661 + 1{,}068\,\kappa_{\mathrm{wt}} - 0{,}609\,\kappa_{\mathrm{wt}}^2 + 0{,}153\,\kappa_{\mathrm{wt}}^3 - 0{,}014\,\kappa_{\mathrm{wt}}^4, \text{if } \zeta_{\mathrm{g}} \ge 0$$

$$C_2 = 0{,}535 + 0{,}426\,\kappa_{\mathrm{wt}} - 0{,}029\,\kappa_{\mathrm{wt}}^2 - (0{,}061 + 0{,}074\,\kappa_{\mathrm{wt}} - 0{,}0085\,\kappa_{\mathrm{wt}}^2)\,\zeta_{\mathrm{g}}, \text{ if } \zeta_{\mathrm{g}} < 0$$

In this paper we give values of $C_1(\kappa_{wt} = 0{,}235)$ for a lot of new moment distributions valid for beam supported at both ends by "forks", with double symmetric cross-section, for which parameter $\zeta_j = 0$ and there is no need to know value of C_3. These C_1 values may be used also for continuous girders being on the safe side. Investigated loading case is shown in Figure 1.

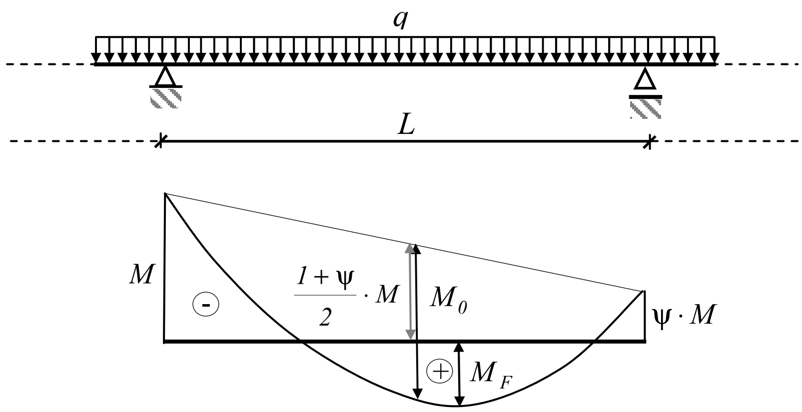


Fig.1: Investigated loading case

The following was taken into account:

$$-1 \le \psi \le 1, \qquad M_0 = \frac{1}{8} q L^2 \ge 0, \qquad M \le 0, \qquad M_F = M_0 + \frac{1+\psi}{2} M + \frac{(1-\psi)^2}{16} \frac{M^2}{M_0} \tag{4}$$

$$|M|_{max} = |M| \quad \text{if} \quad \sqrt{\frac{1+\psi}{2}} - sign(M) \frac{3+\psi}{4} \frac{|M|}{M_0} > 1, \qquad \text{otherwise } |M|_{max} = M_F \tag{5}$$

Elastic critical moment M_{cr} may be calculated from the formula (1) for cases $\zeta_g = 0$. For cases $\zeta_g \neq 0$, because we have not relevant values of factor C_2 for various cases, we can use an approximate value $C_2 = 0{,}5$ for all cases as it was done also in DIN 18 800 (1990, 2008). Then

$$\mu_{cr} \approx \frac{C_1}{k_z} \left[\sqrt{1 + \kappa_{wt}^2 + (0{,}5 \zeta_g)^2} - 0{,}5 \zeta_g \right] \tag{6}$$

The numerical values of factor C_1 were calculated for $\kappa_{wt} = 0{,}235$ and different moment distributions defined by parameters ψ, M_0 / M by an efficient computer program CalcMcr Version 1.9 developed by the authors and they may be used also for the beams with any κ_{wt} values. C_1 values were computed for combination of 21 end moments ratios ψ = -1; -0,9; -0,8; -0,7; -0,6; -0,5; -0,4; -0,3; -0,2; -0,1; 0; 0,1; 0,2; 0,3; 0,4; 0,5; 0,6; 0,7; 0,8; 0,9; 1, and 11 moment ratios M_0 / M = 0; - 0,25; - 0,5; - 0,75; - 1; - 1,25; - 1,5; - 1,75; - 2; -10; - ∞ (see Table 5 and Figure 2). $C_1 \approx 1{,}16$ for M_0 / M = - 10 and $C_1 = 1{,}13$ for $M_0 / M = -\infty$ for all ψ values. Similar table as Table 5 was created also for $0 \le M_0 / M \le \infty$, but it is not given here because of limited size of the paper. Location of the elastic critical moment M_{cr} is identical with location of maximum moment $|M|_{max}$.

Similar tables like Table 1 were created also for four other loadings and boundary conditions:

(i) point load F in the middle of the beam span combined with support moments,

(ii) two point loads F acting in quarters of the beam span combined with support moments,

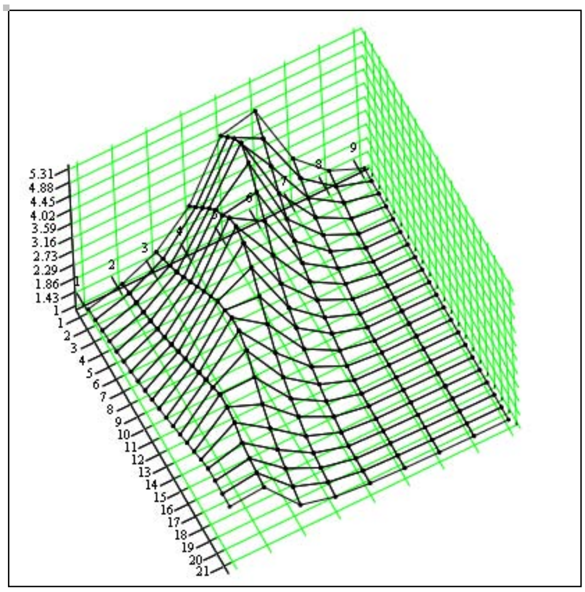
(iii) cantilever under uniform loading q and

(iv) cantilever under tip load F.

Computer program CalcMcr Version 1.9 enables for these 4 loading cases to take into account exactly, according to (1) and (2), also point of load q, F application related to shear center and mono-symmetry of cross-sections.

Tab. 5: Values of factor $C_1(\kappa_{wt} = 0{,}235;\ \psi;\ M_0 / M)$. *Boundary conditions:* $k_y = 1$, $k_z = 1$, $k_w = 1$.

C_1	M_0/M										
ψ	**0**	**- 0,25**	**- 0,5**	**- 0,75**	**- 1**	**- 1,25**	**- 1,5**	**- 1,75**	**- 2**	**-10**	**- ∞**
1	**1,00**	**1,28**	**1,75**	**2,67**	**4,36**	**4,61**	**2,59**	**1,69**	**1,24**		
0,9	**1,06**	**1,37**	**1,93**	**3,03**	**4,70**	**4,14**	**2,36**	**1,58**	**1,23**		
0,8	**1,11**	**1,46**	**2,11**	**3,40**	**5,05**	**3,66**	**2,14**	**1,47**	**1,22**		
0,7	**1,17**	**1,56**	**2,29**	**3,77**	**5,31**	**3,19**	**1,92**	**1,36**	**1,21**		
0,6	**1,24**	**1,68**	**2,52**	**4,11**	**5,07**	**2,78**	**1,75**	**1,26**	**1,21**		
0,5	**1.32**	**1,82**	**2,77**	**4,34**	**4,54**	**2,48**	**1,63**	**1,22**	**1,20**		
0,4	**1,39**	**1,95**	**3,01**	**4,65**	**4,01**	**2,18**	**1,49**	**1,18**	**1,19**		
0,3	**1,48**	**2,11**	**3,26**	**4,77**	**3,34**	**1,98**	**1,39**	**1,20**	**1,19**		
0,2	**1,58**	**2,27**	**3,49**	**4,64**	**2,84**	**1,80**	**1,30**	**1,20**	**1,19**		
0,1	**1,68**	**2,44**	**3,73**	**4,26**	**2,47**	**1,64**	**1,23**	**1,19**	**1,18**		
0	**1,79**	**2,61**	**3,83**	**3,74**	**2,23**	**1,53**	**1,20**	**1,19**	**1,18**	**1,16**	**1,13**
-0,1	**1,90**	**2,79**	**3,99**	**3,26**	**1,96**	**1,39**	**1,18**	**1,18**	**1,17**		
-0,2	**2,03**	**2,96**	**3,97**	**2,76**	**1,79**	**1,29**	**1,19**	**1,18**	**1,17**		
-0,3	**2,15**	**3,11**	**3,75**	**2,38**	**1,63**	**1,23**	**1,19**	**1,18**	**1,17**		
-0,4	**2,28**	**3,28**	**3,29**	**2,10**	**1,49**	**1,21**	**1,19**	**1,18**	**1,17**		
-0,5	**2,40**	**3,27**	**2,84**	**1,92**	**1,40**	**1,21**	**1,19**	**1,17**	**1,17**		
-0,6	**2,54**	**3,18**	**2,47**	**1,70**	**1,30**	**1,20**	**1,19**	**1,17**	**1,16**		
-0,7	**2,65**	**2,93**	**2,19**	**1,56**	**1,25**	**1,21**	**1,19**	**1,17**	**1,16**		
-0,8	**2,65**	**2,83**	**1,99**	**1,49**	**1,25**	**1,21**	**1,19**	**1,17**	**1,16**		
-0,9	**2,64**	**2,77**	**1,81**	**1,43**	**1,26**	**1,21**	**1,19**	**1,17**	**1,16**		
-1	**2,62**	**2,71**	**1,63**	**1,36**	**1,26**	**1,22**	**1,19**	**1,17**	**1,16**		



C_1

Fig. 2: $C_1(\kappa_{wt} = 0{,}235;\ \psi;\ M_0 / M)$ *for* $-1 \leq \psi \leq 1$ *(21 values) and for* $-\infty \leq M_0 / M \leq 0$ *(11 values)*

All above mentioned tables are more general containing both $C_{1,0} = C_1(\kappa_{wt} = 0)$ and $C_{1,1} = C_1(\kappa_{wt} = 1)$ values, what enables to obtain more exact C_1 value for any κ_{wt} by using an interpolation.

3. Design buckling resistance moment $M_{b,Rd}$

The design buckling resistance moment $M_{b,Rd}$ as it is defined in Eurocodes is described in paragraph 3.1. After modifications of Eurocode formulae the influence of plastic reserve and local buckling may be shown. This is done in paragraph 3.2.

3.1 $M_{b,Rd}$ according to Eurocodes EN 1993-1-1 (2005), -1-4 (2006) and EN 1999-1-1 (2007)

A laterally unrestrained member subject to major axis bending should be verified against lateral-torsional buckling as follows

$$\frac{M_{Ed}}{M_{b,Rd}} \leq 1{,}0 \tag{7}$$

where M_{Ed} is the design value of the moment.

The characteristic $M_{b,Rk}$ and design $M_{b,Rd}$ buckling resistance moment of a laterally unrestrained beam should be taken as

$$M_{b,Rk} = \chi_{LT} W_y f_y , \qquad M_{b,Rd} = \frac{M_{b,Rk}}{\gamma_{M1}} \tag{8}$$

where f_y is the yield strength (in EN 1999-1-1 (2007) symbol f_o is used),

γ_{M1} is partial safety factor of material which may be defined in national annex. The recommended values are given in Table 6.

Tab. 6: Recommended values of partial factor γ_{M1}

EN	EN 1993-1-1 (2005)	EN 1993-1-4 (2006)	EN 1993-2 (2006)	EN 1999-1-1 (2007)
γ_{M1}	1,0	1,1	1,1	1,1

The value of reduction factor χ_{LT} for lateral torsional buckling depends on relative slenderness $\bar{\lambda}_{LT}$, and imperfection factor α_{LT}

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta \bar{\lambda}_{LT}^2}}, \qquad \chi_{LT} \leq 1{,}0 \tag{9}$$

$$\Phi_{LT} = 0{,}5\left[1 + \alpha_{LT}(\bar{\lambda}_{LT} - \bar{\lambda}_{LT,0}) + \beta \bar{\lambda}_{LT}^2\right] \tag{10}$$

where $\beta = 1$ in 6.3.2.2 and value $\beta = 0{,}75$ is recommended in 6.3.2.3 of EN 1993-1-1 (2005), (β may be changed in National Annex for rolled I-sections and equivalent I-sections in 6.3.2.3).

α_{LT} is an imperfection factor depending on buckling curve and it is defined in Eurocodes (Table 7),

$\bar{\lambda}_{LT,0}$ is the limit of the horizontal plateau (Table 7),

$\bar{\lambda}_{LT} = \sqrt{\dfrac{W_y f_y}{M_{cr}}}$ is the relative slenderness, (11)

M_{cr} is the elastic critical moment for lateral torsional buckling (see paragraph 2.).

Tab. 7: Values of imperfection factor α_{LT}

EN	Limits		α_{LT}	$\bar{\lambda}_{LT,0}$
EN 1993-1-1 (2005) 6.3.2.2 General case [EN 1993-2 (2006)]	Rolled I-sections	$h/b \leq 2$	0,21	0,2
		$h/b \leq 2$	0,34	
	Welded I-sections	$h/b \leq 2$	0,49	
		$h/b \leq 2$	0,76	
	Other cross-sections		0,76	
EN 1993-1-1 (2005) 6.3.2.3 Rolled I-sections and equivalent I-sections [EN 1993-2 (2006)]	Rolled I-sections	$h/b \leq 2$	0,34	0,4
		$h/b \leq 2$	0,49	
	Welded I-sections	$h/b \leq 2$	0,49	
		$h/b \leq 2$	0,76	
EN 1993-1-4 (2006)	Cold formed sections and hollow sections (welded and seamless)		0,34	
	Welded open sections and other sections for which no test data are available		0,76	0,4
EN 1999-1-1 (2007)	Class 1 and class 2 cross-sections		0,1	0,6
	Class 3 and class 4 cross-sections		0,2	0,4

W_y is the appropriate section modulus as follows:

$W_y = W_{pl,y}$ for Class 1 or 2 cross-sections

$W_y = W_{el,y}$ for Class 3 cross-sections

$W_y = W_{eff,y}$ for Class 4 cross-sections.

Instead of this explicit definition of W_y used in EN 1993-1-1 (2005), the implicit formulae are used in ENV 1993-1-1 (1992) $W_y = \beta_w W_{pl,y}$ and in ENV 1999-1-1 (1998) and in EN 1999-1-1 (2007) $W_y = \alpha_w W_{el,y}$. Shape factors β_w and α_w are defined in Table 8.

Tab. 8: Values of shape factors β_w *in EN 1993-1-1 (2005) and* α_w *in EN 1999-1-1 (2007)*

Cross-section class	EN 1993-1-1 (2005), factor β_w	EN 1999-1-1 (2007), factor α_w	
		Without welds	With longitudinal welds
1	1	W_{pl} / W_{el} *)	$W_{pl,\,haz} / W_{el}$ *)
2	1	W_{pl} / W_{el}	$W_{pl,haz} / W_{el}$
3	W_{el} / W_{pl}	$\alpha_{3,u}$	$\alpha_{3,w}$
4	W_{eff} / W_{pl}	W_{eff} / W_{el}	$W_{eff,haz} / W_{el}$
*) NOTE: These formulae are on the conservative side. For more refined value, recommendations are given in EN 1999-1-1 (2007), Annex F.			

In Table 8 the various section moduli W and $\alpha_{3,u}, \alpha_{3,w}$ are defined as:

W_{pl} plastic modulus of gross section

W_{eff} effective elastic section modulus, obtained using a reduced thickness t_{eff} for the class 4 parts (see 6.2.5.2 in EN 1999-1-1 (2007))

$W_{el,haz}$ effective elastic modulus of the gross section, obtained using a reduced thickness $\rho_{o,haz}t$ for the HAZ material (see 6.2.5.2 in EN 1999-1-1 (2007))

$W_{pl,haz}$ effective plastic modulus of the gross section, obtained using a reduced thickness $\rho_{o,haz}t$ for the HAZ material (see 6.2.5.2 in EN 1999-1-1 (2007))

$W_{eff,haz}$ effective elastic section modulus, obtained using a reduced thickness $\rho_c t$ for the class 4 parts or a reduced thickness $\rho_{o,haz}t$ for the HAZ material, whichever is the smaller (see 6.2.5.2 in EN 1999-1-1 (2007))

$\alpha_{3,u} = 1$ or may alternatively be taken as

$$\alpha_{3,u} = 1 + \frac{\beta_3 - \beta_{mcp}}{\beta_3 - \beta_2} \frac{W_{pl} - W_{el}}{W_{el}} \tag{12}$$

$\alpha_{3,w} = W_{el,haz} / W_{el}$ or may alternatively be taken as

$$\alpha_{3,w} = \frac{W_{el,haz}}{W_{el}} + \frac{\beta_3 - \beta_{mcp}}{\beta_3 - \beta_2} \frac{W_{pl,haz} - W_{el,haz}}{W_{el}} \tag{13}$$

where:

β_{mcp} is the slenderness parameter for the most critical part in the section (see EN 1999-1-1 (2007))

β_2 and β_3 are the limiting values for that same part according to Table 8.

The critical part is determined by the lowest value of $(\beta_2 - \beta)/(\beta_3 - \beta)$.

For aluminium alloys cross-sections without welds it can be written

$$\beta_w = \alpha_w \frac{W_{el}}{W_{pl}}, \quad \alpha_w = \beta_w \frac{W_{pl}}{W_{el}} \tag{14}$$

3.2 Modified formulae of $M_{b,Rd}$ showing plastic reserve and local buckling influence

There are two possible forms how to express the characteristic lateral torsional buckling resistance of member in bending:

a) the form utilising as reference moment the plastic moment resistance of cross section $W_{pl}f_y$ as it is used in EN 1993-1-1 (2005) for steel members and EN 1993-1-4 (2006) for stainless steel members:

$$M_{b,Rk} = \chi_{LT}\beta_W W_{pl} f_y \tag{15}$$

with $$\chi_{LT}\beta_w = \frac{M_{b,Rk}}{W_{pl}f_y} = \frac{\beta_w}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta(\sqrt{W_{pl}f_y / M_{cr}}\sqrt{\beta_W})^2}}, \quad \text{if} \quad \sqrt{\frac{W_{pl}f_y}{M_{cr}}} \le \frac{\bar{\lambda}_{LT,0}}{\sqrt{\beta_W}} \tag{16}$$

otherwise $$\chi_{LT}\beta_W = M_{b,Rk} / W_{pl}f_y = \beta_W \tag{17}$$

where $$\Phi_{\mathrm{LT}} = 0{,}5\left[1+\alpha_{\mathrm{LT}}\left(\sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}} - \bar{\lambda}_{\mathrm{LT,0}}\right)+\beta\left(\sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}}\right)^2\right] \tag{18}$$

with $\beta_{\mathrm{w}} = 1$ for class 1, 2, $\beta_{\mathrm{W}} = W_{\mathrm{el}}/W_{\mathrm{pl}}$ for class 3 sections and $\beta_{\mathrm{W}} = W_{\mathrm{eff}}/W_{\mathrm{pl}}$ for class 4 sections.

b) the form utilising as reference moment the elastic moment resistance of cross section $W_{\mathrm{el}}f_{\mathrm{y}}$ as it is used in EN 1999-1-1 (2007) for design of aluminium members:

$$M_{\mathrm{b,Rk}} = \chi_{\mathrm{LT}}\alpha_{\mathrm{W}}W_{\mathrm{el}}f_{\mathrm{y}} \tag{19}$$

with $$\chi_{\mathrm{LT}}\alpha_{\mathrm{w}} = \frac{M_{\mathrm{b,Rk}}}{W_{\mathrm{el}}f_{\mathrm{y}}} = \frac{\alpha_w}{\Phi_{\mathrm{LT}} + \sqrt{\Phi_{\mathrm{LT}}^2 - \beta\left(\sqrt{W_{\mathrm{el}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\alpha_{\mathrm{W}}}\right)^2}}, \quad \text{if} \quad \sqrt{\frac{W_{\mathrm{el}}\,f_{\mathrm{y}}}{M_{\mathrm{cr}}}} \le \frac{\bar{\lambda}_{\mathrm{LT,0}}}{\sqrt{\alpha_{\mathrm{W}}}} \tag{20}$$

otherwise $$\chi_{\mathrm{LT}}\alpha_{\mathrm{W}} = M_{\mathrm{b,Rk}}/W_{\mathrm{el}}f_{\mathrm{y}} = \alpha_{\mathrm{W}} \tag{21}$$

where $$\Phi_{\mathrm{LT}} = 0{,}5\left[1+\alpha_{\mathrm{LT}}\left(\sqrt{W_{\mathrm{el}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\alpha_{\mathrm{W}}} - \bar{\lambda}_{\mathrm{LT,0}}\right)+\beta\left(\sqrt{W_{\mathrm{el}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\alpha_{\mathrm{W}}}\right)^2\right] \tag{22}$$

with $\alpha_{\mathrm{w}} = W_{\mathrm{pl}}/W_{\mathrm{el}}$ for class 1, 2, $\alpha_{\mathrm{W}} = 1$ for class 3 sections and $\alpha_{\mathrm{W}} = W_{\mathrm{eff}}/W_{\mathrm{el}}$ for class 4 sections.

This modification may be done for formulae of all methods defining lateral buckling curves used in EN 1993-1-1 (2005): (i) general case in 6.3.2.2, with or without utilising 6.3.2.2(4), which means that $\chi_{\mathrm{LT}} = 1$ in interval $0 \le \bar{\lambda}_{\mathrm{LT}} \le \bar{\lambda}_{\mathrm{LT,0}}$, (ii) rolled I-sections or equivalent welded sections in 6.3.2.3 with or without utilising factor f defined in 6.3.2.3(2).

According to 6.3.2.3(2) of EN 1993-1-1 (2005) the design buckling resistance moment may be increased by dividing by factor f, which may be defined in National Annex. The following minimum values are recommended in EN 1993-1-1 (2005):

$$f\left(k_{\mathrm{c}}, \sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}}\right) = 1 - 0{,}5(1-k_{\mathrm{c}})\left[1 - 2\left(\sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}} - 0{,}8\right)^2\right] \tag{23}$$

but $$f\left(k_{\mathrm{c}}, \sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}}\right) \le 1{,}0 \tag{24}$$

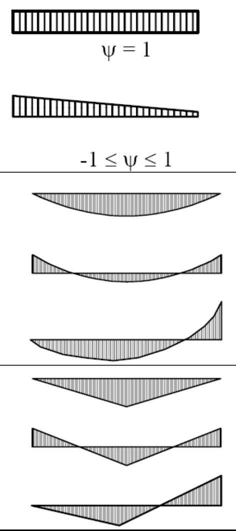
The factor f defined by (23) is smaller than 1 only in the interval

$$\max\left[\bar{\lambda}_{\mathrm{LT,0}};\, 0{,}1\right] \approx \max\left[\bar{\lambda}_{\mathrm{LT,0}};\, 0{,}8+\sqrt{0{,}5}\right] \le \sqrt{W_{\mathrm{pl}}f_{\mathrm{y}}/M_{\mathrm{cr}}}\sqrt{\beta_{\mathrm{W}}} \le 0{,}8+\sqrt{0{,}5} \approx 1{,}5 \tag{25}$$

The correction factor k_{c} for different moment distributions is given in Table 9a and in Table 6.6 in EN 1993-1-1 (2005)). In corrigendum EN 1993-1-1 (2005) from April 2009 it is recommended to use k_{c} values for calculation of C_1 values used in EN 1993-1-1 (2005) in Table A.1, formulae BB.5 and BB.9. The comparisons in Table 9b show that C_1 values calculated from k_{c} values are only approximate ones. It is better to use more exact C_1 values of authors published in Baláž, I. – Koleková, Y. (2000 b), in EN 1999-1-1 (2007) or in STN EN 1993-1-1/NA (2007).

It is very important to mention that all k_{c} values compared in Table 9b relate to the cross-section in the middle of the span and they are valid for boundary conditions $k_{\mathrm{z}} = 1$ (both beam ends are restrained against lateral movement and free to rotate in plan) and $k_{\mathrm{w}} = 1$ (both beam ends are restrained against rotation about longitudinl axis and free to warp).

Tab. 9a: k_c from EN 1993-1-1 (2005)

Moment distribution	k_c
 $\psi = 1$	1,0
$-1 \le \psi \le 1$	$\frac{1}{1{,}33 - 0{,}33\psi}$
	0,94
	0,90
	0,91
	0,86
	0,77
	0,82

Tab. 9b: Comparison of C_1 values of authors with k_c^{-2}

k_c	$1/k_c^2 \approx C_1$	$C_{1,0} = C_1(\kappa_{wt} = 0)$	$C_{1,1} = C_1(\kappa_{wt} = 1)$
1 $\psi = 1$	1	1	1
0,924 $\psi = 0{,}75$	1,172	1,139	1,141
0,858 $\psi = 0{,}5$	1,357	1,312	1,320
0,802 $\psi = 0{,}25$	1,556	1,522	1,551
0,752 $\psi = 0$	1,769	1,770	1,847
0,708 $\psi = -0{,}25$	1,995	1,753	2,027
0,669 $\psi = -0{,}5$	2,235	2,331	2,591
0,634 $\psi = -0{,}75$	2,489	2,547	2,852
0,602 $\psi = -1$	2,756	2,555	2,733
0,94	1,132	1,127	1,132
0,90	1,235	0,5*2,576=1,288	0,5*2,608=1,304
0,91	1,208	-	-
0,86	1,352	1,348	1,363
0,77	1,687	1,683	1,726
0,82	1,487	-	-

*) Values 2,576 and 2,608 relate to $M_{\max}$ and 0,9, 1,235, 1,288 and 1,304 relate to M in midspan

Distribution of modified lateral torsional buckling curves defined in 6.3.2.2 with utilising 6.3.2.2(4) is shown in Fig.3. The ends of plateaux are denoted by relative slenderness values $\bar{\lambda}_{\mathrm{LT},0} = 0.4$, $\bar{\lambda}_{\mathrm{LT},0} / \sqrt{0.85} = e = 0{,}434$ and $\bar{\lambda}_{\mathrm{LT},0} / \sqrt{0.7} = f = 0{,}478$ (Fig.3). Note the discrepancies in member resistances ($M_{\mathrm{el}} > M_{\mathrm{pl}}$) at these points when 6.3.2.2(4) is utilised.

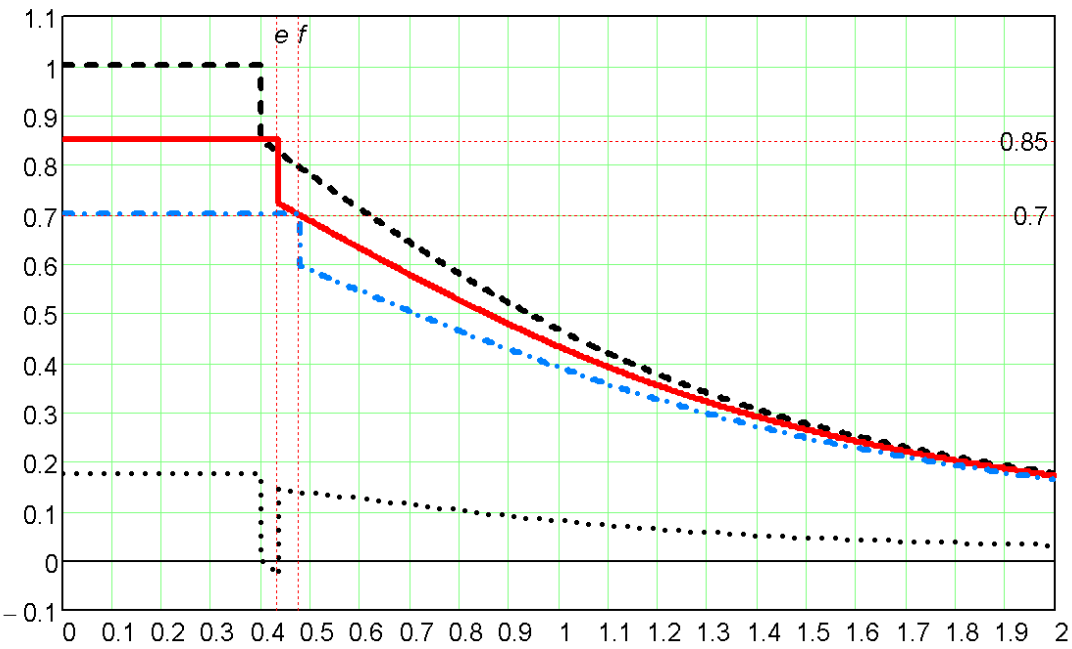


Fig. 3: Functions $\chi_{\mathrm{LT}}\beta_{\mathrm{W}} = f(\sqrt{W_{\mathrm{pl}} f_{\mathrm{y}} / M_{\mathrm{cr}}})$ based on EN 1993-1-1 (2005), 6.3.2.2 calculated for imperfection factor $\alpha_{\mathrm{LT}} = 0{,}76$, relative slenderness defining end of the plateau $\bar{\lambda}_{\mathrm{LT},0} = 0{,}4$, $\beta = 1$ and for (i) $\beta_{\mathrm{W}} = 1$ (black dashed line), (ii) $\beta_{\mathrm{W}} = W_{\mathrm{el}} / W_{\mathrm{pl}} = 0{,}85$ (red solid line) and (iii) $\beta_{\mathrm{W}} = W_{\mathrm{eff}} / W_{\mathrm{pl}} = 0{,}7$ (blue dot-and-dashed line). Plastic reserve of member is defined by the ordinates of the bottom black dotted line, which should be multiplied by 100 to obtain plastic reserve in %.

4. Conclusions

The paper is devoted to (i) critical moment M_{cr} and (ii) lateral torsional buckling resistance moment $M_{b,Rd}$ of metal (steel and aluminium) beams.

The original results of the authors are presented concerning calculation of M_{cr} by using approximate formulae convenient for standardization and educational purposes and for engineering practice. The results are based on large parametrical studies (Baláž, I., 1999-2001, 2005, 2007, Baláž, I. – Koleková, Y., 1999-2001, 2002, Koleková, Y., 1999) which showed that procedure used in prestandard Eurocodes ENV 1993-1-1 (1992) and ENV 1999-1-1 (1998) may lead in many cases to incorrect results. Authors results, which were the first time published in (Baláž, I. – Koleková, Y., 1999 a, b, 2000 b, c, 2002 a, b) are today used in many international and national standardization documents including Eurocodes (Belgian National Annex, 2005, Czech National Annex ČSN EN 1993-1-1, 2006, Austrian National Annex ÖNORM B 1993-1-1, 2007, Slovak National Annex STN EN 1993-1-1/NA, 2007, Design Manual For Structural Stainless Steel, 2006, Deutscher Ausschluß für Stahlbau, 2005, ECCS Technical Committee 8 – Stability, 2006, Excerpt from the Background Document to EN 1993-1-1, 2010, EN 1999-1-1, 2007). The correctness and the exactness of results based on authors results (Table 1-4) were verified in habilitation thesis (Koleková, Y., 1999) and later also in two independent Brazilian and Slovak PhD thesis (Fruchtengarten, J., 2005, Živner, T., 2010). The Brazilian PhD thesis stated that M_{cr} values calculated on the basis of the authors results are the best among all used approximate formulae. Procedure of M_{cr} calculation based on authors results were introduced in the engineering practice of many countries. In Slovakia it was thanks to courses for the engineers in practice organized by Universities and Slovak Chamber of Civil Engineers and the textbooks written for them (Baláž, I., 2007, 2010, 2012) and for the university students. It was also showed how the procedure used for many years in Czechoslovak, Czech and Slovak standards may be improved (Baláž, I., 1980, 1997, 1998, 2000), Baláž, I. – Živner, T., 2007). The detailed numerical examples were published in the above mentioned textbooks and in the several papers (e.g. Baláž, I. (2012). The authors created for engineers in practice and for students at Universities the original computer program CalcMcr.

New C_1 values derived by authors for uniform loading combined with end moments are presented in Table 5 and Figure 2.

The Table 9a, b shows that using of k_c values recommended in corrigendum EN 1993-1-1 (2005) from April 2009 for calculation of C_1 values may lead to approximate values, which may be used in correct way only if user knows that: (i) relating boundary conditions are $k_z = 1$ (both beam ends are restrained against lateral movement and free to rotate in plan) and $k_w = 1$ (both beam ends are restrained against rotation about longitudinal axis and free to warp and that (ii) k_c values compared in Table 9b with authors more exact C_1 values are valid for M in midspan and not to cross-section were M_{max} is located.

The paper presents also the way how the Eurocode formulae (EN 1993-1-1, 2005), EN 1993-1-4, 2006), EN 1999-1-1, 2007) for calculation of lateral torsional buckling resistance $M_{b,Rd}$ may be modified to show clearly influence of plastic reserve and local buckling on beam resistance (paragraph 3.2 and Figure 3). This may be very useful for engineers in practice. The much more similar diagrams as it is on Figure 3 were published in (Baláž, I. – Koleková, Y., 2007, 2008, 2009).

The paper is devoted to the lateral torsional buckling resistance of metal (steel, stainless steel, aluminium) beams, but the presented results may be used also for design and verification of structures made of other structural materials (timber, concrete, composite steel and concrete structures). The results relating to lateral torsional buckling of: (i) timber beams authors solved in the papers (Baláž, I., 2001, 2005, Baláž, I. – Koleková, Y., 2004 a, b), (ii) concrete beams in (Baláž, I. – Živner, T., 2006), (iii) aluminium beams in (Baláž, I. – Valach, P., 1997, Baláž, I. – Koleková, Y. –Ároch, R., 1998, Baláž, I. – Koleková, Y., 2000 a, 2007, 2008). The influence of beam end stiffeners was solved in (Živner, T., 2010, Živner, T. – Baláž, I., 2010).

The second author is the member of 5 working Evolution Groups: EG EN 1993-1-1, EG EN 1993-1-3, EG EN 1993-1-5, EG EN 1993-2 and EG EN 1999-1-1. All members of EGs are very active without any financial support on the European level. Even accommodation, travel, food and other expenses must be covered by own budget of EGs members. EGs are responsible for maintenance of existing Eurocodes (creating of Corrigenda and Amendments) and for further development of the next generation of Eurocodes. Nobody else from Slovak republic is member of EGs.

We spent blessed moments in investigation of these problems during several years despite of the fact that financial support was very poor. The reasons of poor funding were: a) the funding into the science in Slovakia is for long time deeply undersized comparing with all other EU countries, b) only the results published in current journals are highly evaluated even when they are without any useful application, c) journals with high impact factors are preferred despite of the fact that the European Association of Science Editors already in November 2007 issued an official statement recommending "that journal impact factors are used only – and cautiously – for measuring and comparing the influence of entire journals, but not for the assessment of single papers, and certainly not for the assessment of researchers or research programs", d) a well known specialist from the Institute of Construction and Architecture, Slovak Academy of Sciences in Bratislava evaluated our first grant application for this project in 1999 and in his review he gave us the lowest possible grades and used the following wordings: "investigators – underaverage; scientific team composition – inadequate; expected contributions – not important; scientific goals – obsolete, everything was already solved".

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