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# INTRODUCTION OF THE ANALYTICAL TURBULENT VELOCITY PROFILE BETWEEN TWO PARALLEL PLATES

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**Abstract:** *A new analytical velocity profile between two parallel plates is introduced in this article. It is possible to use this velocity profile for both laminar and turbulent flow. All necessary parameters can be obtained from the unit flow rate and the pressure drop. We can also use this model in case when the material of the upper and lower wall is different.*

**Keywords:** ***velocity profile, laminar flow, turbulent flow, flow between parallel plates.***

## 1. Introduction

The laminar velocity profile between two laminar plates is very well known from the fundamental lectures of the hydromechanics. The velocity profile, in case of Poiseuille flow means the laminar flow governed by the pressure gradient, can be then expressed by the quadratic function.

$$v_{(x)} = \frac{\Delta p}{L.\mu} \cdot \frac{h^2}{2}\left[1 - \left(\frac{x_2}{h}\right)^2\right] \tag{1}$$

Where $\Delta p/L$ is a pressure gradient, $\mu$ is dynamic viscosity, h is a half of distance between the parallel plates.

Maximal velocity is in the middle of the channel where y=0.

$$v_{(x\max)} = \frac{\Delta p}{L.\mu} \cdot \frac{h^2}{2} \tag{2}$$

Then the velocity profile could be expressed as

$$v_{(x)} = v_{(x\max)}\left[1 - \left(\frac{y}{h}\right)^2\right] \tag{3}$$

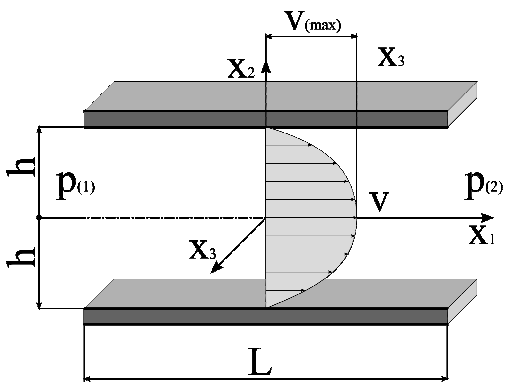


*Fig. 1.Velocity profile between two parallel plates*

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It is more complicated to find any analytical solution in the case of turbulent flow. There is some expression which was found on the basis of experimental data. It is known as the *power law*. Unfortunately it was derived only for the flow in a pipe with radius R.

$$v_{(x)} = v_{(x\,max)} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{n_0} \tag{4}$$

Where r is a radius within interval <0,R>, R is the pipe radius, $n_0$ is the exponent which is a function of the Reynolds number. This exponent can be evaluated from the expression

$$\frac{1}{n_0} = 1 + \sqrt[6]{\frac{Re}{50}} \tag{5}$$

There is some discrepancy in this power law definition. It cannot be used near the wall, for r=R, because the derivative has an infinite value there. The wall shear stress $\tau_w$ is then infinite, which does not correspond with the reality.

There is also another power law definition by Munson (2006).

$$v_{(x)} = v_{(x\,max)} \left( 1 - \frac{r}{R} \right)^{1/n_0} \tag{6}$$

Where $n_0$ is a function of the Reynolds number, see Munson (2006). This turbulent velocity profile has problems in two locations. The first one is the same as in the previous formulation. The wall shear stress goes to infinity. The second problematic location is in the middle of the channel, the derivative for r=0 is not zero.

From the above examples it is clear that these models have problems near the wall. Therefore it is necessary to focus on the near wall region – boundary shear layer. The boundary shear layer in the case of turbulent flow can be divided into three regions.

The first region is called viscous sub-layer. This region is close to the wall and the viscosity plays a dominant role in this region. The velocity profile is linear within this region.

The second region is the transition area – this is the region of smooth transition into the logarithmic velocity profile.

The third one is the region where the turbulent viscosity plays a dominant role.

$$\frac{v_{(x)}}{v^*} = \frac{1}{\kappa}.\log\frac{v^*.y}{\nu} + B \tag{7}$$

We can define dimensionless velocity $v^+$ and dimensionless distance from the wall $y^+$.

$$v^+ = \frac{v_{(x)}}{v^*} \tag{8}$$

$$y^+ = \frac{v^*.y}{\nu} \tag{9}$$

Where shear velocity v* is expressed by the formula

$$v^* = \sqrt{\frac{\tau_w}{\rho}} \tag{10}$$

Many different authors use different values of constants κ and B. For example Janalík (2008) uses the κ=0,174 and B=5,5 or Munson (2006) uses κ=0,4 and B=5 or Pope (2008) uses κ=0,41 and B 5,2.

## 2. Derivation of a new velocity profile

The new velocity profile is derived on the basis of the vorticity distribution over the space between two parallel plates. This velocity profile can be used for all, laminar, turbulent and constant velocity profile. The derivation of this type of velocity profile is based on a Biott-Savart law, the derivation of it is made in Brdička 2000, applied on a straight vortex filament with circulation Γ.

$$v_i = \frac{\Gamma}{2.\pi}.\frac{1}{r^2}.\varepsilon_{i3k}.(x'_k - x_k) \tag{11}$$

This expression is valid for a special situation when the straight vortex filament is parallel to the $x_3$ axis. This is also an expression of the velocity induced by a single potential vortex in case of 2D flow (Lewis 1991). Einstein summation convection is used in the expression (11). The $v_i$ is the velocity induced by the single vortex, the Γ is a circulation around the single vortex, $x_k$ – are coordinates of vortex location, $x'_k$ are coordinates of induced velocity location, $\varepsilon_{ijk}$ is an Levi-Civita tensor of 3rd order, r is a distance between vortex and point of induced velocity.

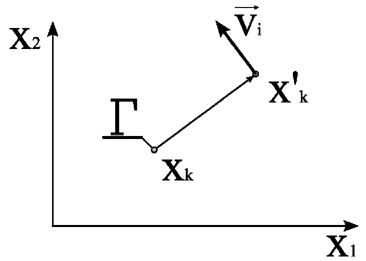


*Fig. 2.Velocity induced by a single 2D vortex.*

### Velocity induced by a plain vortex sheet

There is another well known example of the velocity induced by an infinite plain vortex sheet Lewis (1991). This sheet consists of the vortex filaments, with constant circulation Γ which are parallel to the $x_3$ axis. The situation is depicted in the Fig. 3. It is necessary to define linear vorticity density γ in this case. The circulation dΓ around the infinitesimal length of sheet ds can be expressed this way

$$d\Gamma = \gamma.ds \Rightarrow \gamma = \frac{d\Gamma}{ds} \tag{12}$$

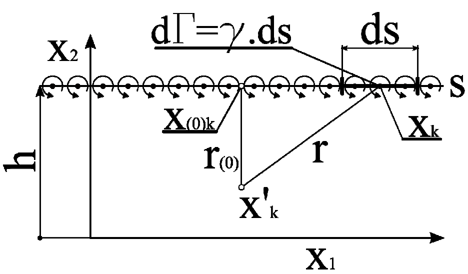


*Fig. 3.Velocity induced by the plain vortex sheet parallel to the $x_1$,$x_3$ plain.*

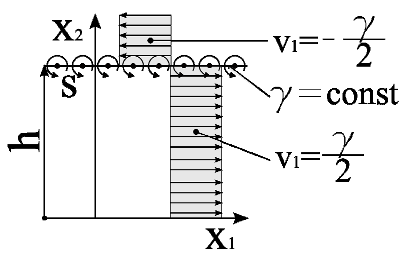


*Fig.4. Velocity profile induced by a plain vortex sheet parallel to the $x_1$,$x_3$ plain.*

In case that the vorticity density is constant, over the whole sheet, the velocity induced by a vortex sheet can be expressed this way.

$$v_i = \frac{\gamma}{2}.\varepsilon_{i3k}.\frac{\left(x'_k - x_{(0)k}\right)}{r_{(0)}} \tag{13}$$

Where γ is the vorticity density, $x'_k$ are the coordinates of location where the velocity is induced, $x_{(0)k}$ are the coordinates of $x'_k$ point projection on the vortex sheet, $r_{(0)}$ is a distance of point $x'_k$ from the vortex sheet. For the case depicted in a fig. 3, when the vortex sheet is collinear with plane $x_1$ $x_3$ , the velocity components are:

$$v_1 = -\frac{\gamma}{2};\ v_2 = 0 \ \text{ within the interval } x'_2 \in \left(h, \quad +\infty\right) \tag{14}$$

$$v_1 = \frac{\gamma}{2};\ v_2 = 0 \ \text{ within the interval } x'_2 \in \left(h, \quad -\infty\right) \tag{15}$$

This type of flow is depicted in the fig 4. It means that velocity within half-plane above vortex sheet is constant, parallel to the vortex sheet and with the orientation to the left. Velocity within the half-plane below vortex sheet is also parallel to the vortex sheet but with the orientation to the right.

**Velocity induced by two vortex sheets with the opposite vorticity density orientation**

Now it is possible to study fluid flow with two parallel vortex sheet $s_{(1)}$, and $s_{(2)}$. The magnitude of the vorticity density of both vortex sheets is equal but the orientation is opposite. It means that $\gamma_{(1)}=\gamma$ and $\gamma_{(2)}=-\gamma$. The vortex sheet $s_{(1)}$ position is $x_2=h_1$. The vortex sheet $s_{(2)}$ position is $x_2=-h_2$. This situation is depicted in the fig. 5.

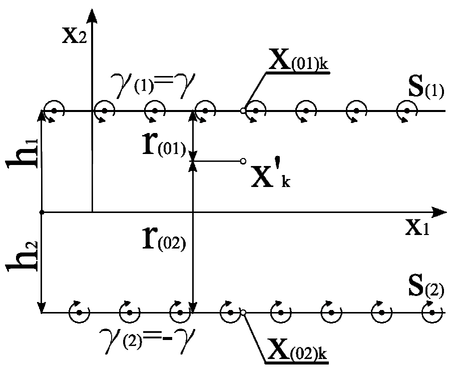


*Fig. 5.Velocity induced by the two plain vortex sheets parallel to the $x_1$,$x_3$ plain.*

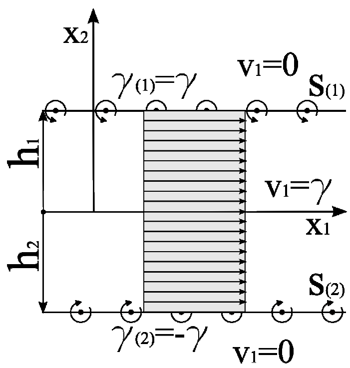


*Fig.6. Velocity profile induced by a plain vortex sheet parallel to the $x_1$,$x_3$ plain.*

The induced velocity for this case should be expressed by this formula

$$\mathrm{v}_i = \sum_{j=1}^{n} \frac{\gamma_{(j)}}{2}.\varepsilon_{i3k}\frac{\left(x'_k - x_{(0j)k}\right)}{r_{(0j)}} \tag{16}$$

This formula is general for an arbitrary number n of vortex sheets. It is possible, for case of two vortex sheets with respect to the (16), to write

$$v_i = \frac{\gamma}{2}.\varepsilon_{i3k}\frac{\left(x'_k - x_{(01)k}\right)}{r_{(01)}} - \frac{\gamma}{2}.\varepsilon_{i3k}\frac{\left(x'_k - x_{(02)k}\right)}{r_{(02)}} \tag{17}$$

Explanation all parameter in above equation are clear from the fig. 5.

It is possible to have three different solutions now.

$$v_1 = 0\,;\;\; v_2 = 0 \quad \text{within the interval } x'_2 \in (h, \quad +\infty) \tag{18}$$

$$v_1 = \gamma\,;\;\; v_2 = 0 \quad \text{within the interval } x'_2 \in (-h, \quad +h) \tag{19}$$

$$v_1 = 0\,;\;\; v_2 = 0 \quad \text{within the interval } x'_2 \in (-\infty, \quad -h) \tag{20}$$

This solution is very nice and reasonable. It is kind of the fluid flow between two parallel plates for infinite Reynolds number Re=∞.

Now it is only a small step to find the expression of the velocity profile for the finite Reynolds number.

**Velocity profile for the continuous vorticity distribution over the cross-section**

It has to be assumed, for this case, that the vorticity density is not the linear density but it is the planar vorticity density. It will be function of the $x_2$ coordinate between the plates. Vorticity density for the fixed coordinate $x_2$ is constant. The situation is depicted in a fig 7. It is necessary to define the coordinate system. The axis $x_1$ is parallel to the plates and it is located in the maximal velocity position.

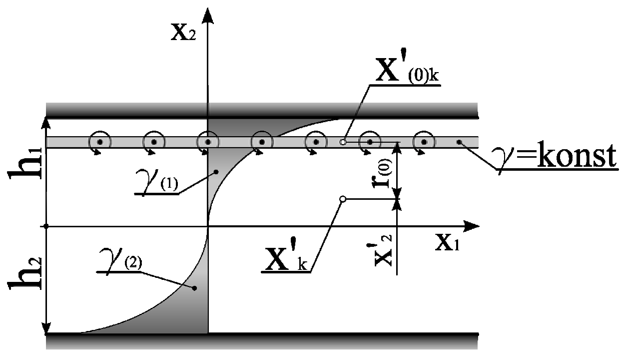


*Fig. 7.The distribution of the planar vorticity density between two plates*

The velocity profile is derived under these assumptions:

- It is the fluid flow between two parallel infinite plane plates.
- The vorticity density distribution is continuous between plates. It will be described by two polynomial functions $\gamma_{(1)}$ and $\gamma_{(2)}$ of the N[th] order.
- The unit flow rate Q and the pressure drop Δp are known parameters.

The continuous vorticity density distribution can be expressed by the next formulas.

$$\gamma_{(1)} = \sum_{n=0}^{N} A_{(n)}.|x_2|^n \tag{21}$$

$$\gamma_{(2)} = -\sum_{n=0}^{N} B_{(n)}.|x_2|^n \tag{22}$$

Now the velocity can be expressed this way

$$v_{(+)1} = \sum_{n=0}^{N} \frac{1}{2.(n+1)} \left( B_{(n)}.h_2^{n+1} + A_{(n)}.h_1^{n+1} - 2.A_{(n)}.|x_2|^{n+1} \right) \text{ within the interval } x_2 = \langle 0, \quad h_1 \rangle \tag{23}$$

$$v_{(-)1} = \sum_{n=0}^{N} \frac{1}{2.(n+1)} \left( B_{(n)}.h_2^{n+1} + A_{(n)}.h_1^{n+1} - 2.B_{(n)}.|x_2|^{n+1} \right) \text{ within the interval } x_2 = \langle -h_2, \quad 0 \rangle \tag{24}$$

It is necessary to determine the coefficients $A_{(n)}$ and $B_{(n)}$ for n=1-N. These coefficients can be determined on the basis of the following conditions.

- Slip condition on the walls
  $v_{(+)} = 0$, for $x_2 = h_1$
  $v_{(-)} = 0$, for $x_2 = -h_2$
- The unit flow rate is known.
- Circulation around element $dx_1.(h_{(1)}+h_{(2)})$ is zero.
- The necessary number of derivative at point $x_2 = 0$ is zero.

On the basis of these conditions it is possible to express velocity profile

$$v_{(+)1} = \frac{(N+2)}{(N+1)}.\frac{Q}{(h_1+h_2)}.\left[1-\left(\frac{|x_2|}{h_1}\right)^{N+1}\right] \text{ within the interval } x_2 = \langle 0, \quad h_1 \rangle \tag{25}$$

$$v_{(-)1} = \frac{(N+2)}{(N+1)}.\frac{Q}{(h_1+h_2)}.\left[1-\left(\frac{|x_2|}{h_2}\right)^{N+1}\right] \text{ within the interval } x_2 = \langle -h_2, \quad 0 \rangle \tag{26}$$

The maximal and average velocity can be expressed this way

$$v_{(max)1} = \frac{(N+2)}{(N+1)}.\frac{Q}{(h_1+h_2)} \tag{27}$$

$$v_{(av)1} = \frac{Q}{(h_1+h_2)} \tag{28}$$

The expression for the number N has to be found now. It is possible to do this from the known pressure drop.

$$N = \frac{h_1.h_2}{\mu.v_{(av)}}.\frac{(p_1-p_2)}{L} - 2 \tag{29}$$

Where $\mu$ is the dynamic viscosity. All above expressions are derived for a case that the plate 1 and plate 2 were made from different materials. So it means that there could be different conditions on the walls. If the conditions on the plates are identical then the velocity profile will be symmetrical ($h_1=h_2 = h$) and it is possible to write

$$v_1 = \frac{(N+2)}{(N+1)}.\frac{Q}{2.h}.\left[1-\left(\frac{|x_2|}{h}\right)^{N+1}\right] \text{ within the interval } x_2 = \langle -h, \quad h \rangle \tag{30}$$

$$N = \frac{h^2}{\mu.v_{(av)}}.\frac{(p_1-p_2)}{L} - 2 \tag{31}$$

This expression can be used for all types velocity profiles from laminar velocity profile (N=1), for turbulent velocity profiles and also for a piston velocity profiles $N\to\infty$. There is no problem with the derivatives near the wall and with the zero value of the first derivative in the centre of channel.

## 3. Discussion

The formal comparison of different velocity profiles will be presented in this chapter. The only problem of it is that the power law velocity profiles are derived for a turbulent flow in pipes. But it is possible to compare the velocity profiles normalized by $v_{max}$ velocity. Comparison of velocity profiles is in the fig. 8. It is only formal comparison because the pressure drop or the friction factor for the fluid flow between the parallel plates is not known. Therefore the three new velocity profiles for different values of power N are compared with Munson (2006) power law velocity profile and with Janalík (2008) power law velocity profile in the fig 8. It is apparent that the character all kind of velocity profiles are different. The Janalik's velocity profile is rather rounded even for very high Reynolds number. The Munson velocity profile has no continuous first derivative in the center of channel. The new velocity profiles have also a problem that there is a zero second derivative in the centre of channel. It means that there is an infinite radius there. But this problem should be solved during derivation its derivation. It means that the velocity profile can be modified in way to ensure non zero second derivative. But it has not been done yet.

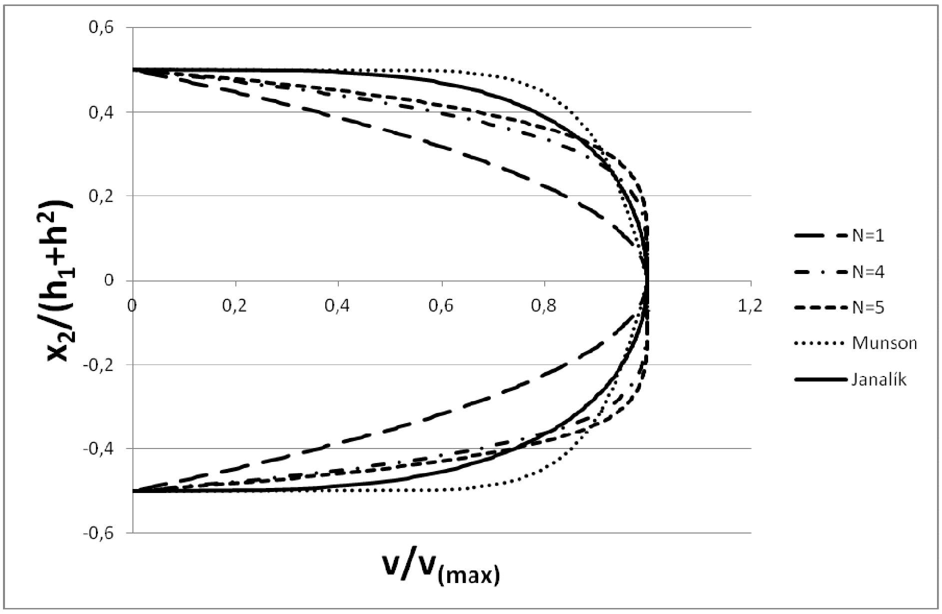


*Fig. 8.New velocity profiles in comparison with power law velocity profiles.*

It is possible also compare the velocity profile near the wall with a logarithm wall law. This comparison is depicted in a fig. 9.

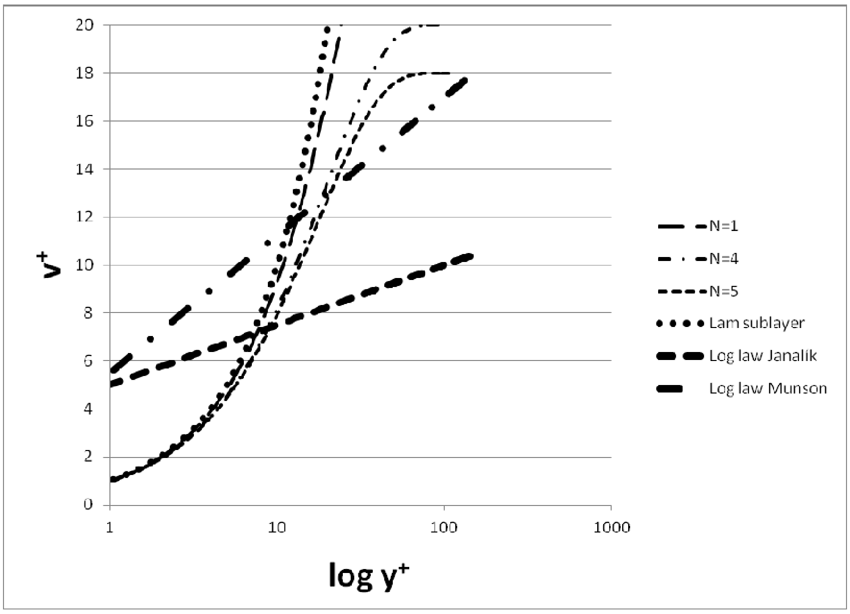


*Fig. 9.The new velocity profiles comparison with log wall law.*

The comparison is not so good because the exponent N is not evaluated, because there is no a measuring of pressure drop and flow rate. Therefore there are depicted several velocity profiles for different exponent N. It is apparent that the boundary shear layer in case of the new velocity profile is thinner than in the case of real velocity profile. This is probably consequence of the zero value of the second derivative of the velocity profile in the channel centre. There is no comparison for the power law profiles because there is no possible to express $y^+$ and $v^+$ because infinite shear stress at the wall.

At the end it would be interesting to draw the shear stress over channel. It is known that the total shear stress is linear. Total shear stress is a sum of the viscous shear stress $\tau_\mu$ and the turbulent or (Reynolds) shear stress $\tau_t$.

$$\tau = \tau_\mu + \tau_t \tag{32}$$

Viscous shear stress should be expressed from a velocity profile formula (29).

$$\tau_\mu = \mu . v_{(max)} . \frac{(N+1)}{h} . \left(\frac{|x_2|}{h}\right)^N \text{ within the interval } x_2 = \langle -h, \quad 0 \rangle \tag{33}$$

$$\tau_\mu = -\mu . v_{(max)} . \frac{(N+1)}{h} . \left(\frac{|x_2|}{h}\right)^N \text{ within the interval } x_2 = \langle 0, \quad h \rangle \tag{34}$$

This stress can be expressed in a dimensionless form

$$\frac{\tau_\mu . h}{\mu . v_{(max)}} = (N+1) . \left(\frac{|x_2|}{h}\right)^N \text{ within the interval } x_2 = \langle -h, \quad 0 \rangle \tag{35}$$

$$\frac{\tau_\mu . h}{\mu . v_{(max)}} = -(N+1) . \left(\frac{|x_2|}{h}\right)^N \text{ within the interval } x_2 = \langle 0, \quad h \rangle \tag{36}$$

The viscous stress together with the Reynolds stress for three different exponents N are depicted in the fig. 10. This also can't be compared with the power law velocity profiles because its shear stress at the wall is infinite.

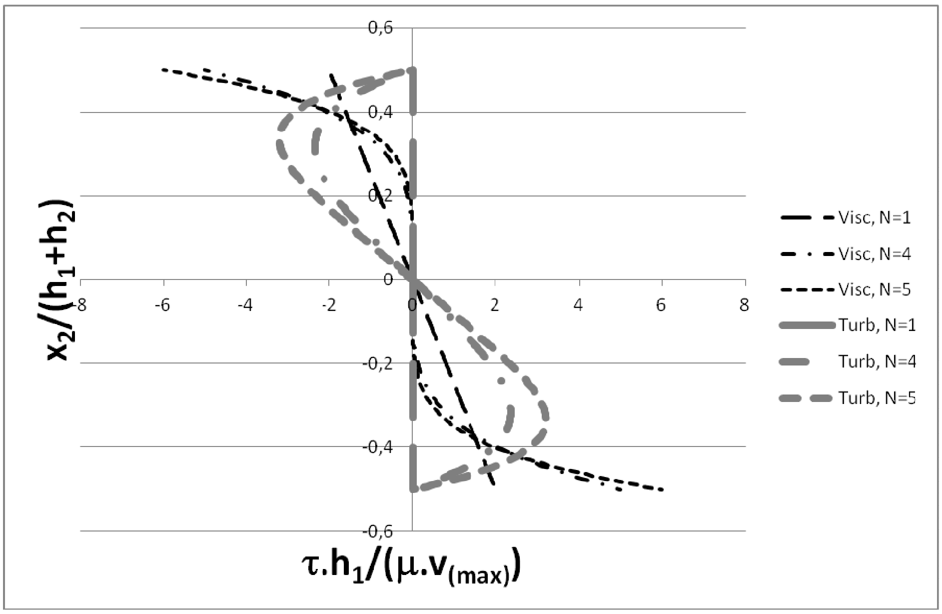


*Fig. 10.The Reynolds and viscous stresses derived from a new velocity profile..*

## 4. Conclusion

The new velocity profile based on a vorticity distribution between two parallel plates has been presented in this paper. This velocity profile is better than the power law velocity profiles because it has not infinite shear stress at the wall. It means that it is possible to express the Reynolds stresses and it is possible to compare this profile with the logarithmic law near the wall. This is not possible in case of the power law velocity profiles because they have infinite derivative near the wall. The new velocity profile has only one problem which can be removed. The problem is that this profile has zero second derivative in the centre of channel. It means that there is an infinite radius of curvature. It is also necessary to compare this velocity profile directly with an experimental velocity profiles. This work can help in understanding or even in modeling of boundary shear layers in CFD software.

## List of Symbols

| Symbol | Units | Description |
|---|---|---|
| $A_{(n)}$ | Varies | Polynom coefficients |
| $B_{(n)}$ | Varies | Polynom coefficients |
| $h$ | [m] | Half distance between two parallel plates |
| $h_1, h_2$ | [m] | Distance of plate $s_1/s_2$ from a coordinate system origin |
| $L$ | [m] | Length. Distance between the pressure location measuring. |
| $n_0$ | - | Exponent |
| $\Delta p$ | [Pa] | Pressure difference |
| $p_{(1)}, p_{(2)}$ | [Pa] | Pressure at location 1 or 2 respectively |
| $Q$ | [m] | Unit flow rate. Flow rate between two parallel plates with 1m width. |
| $r, R$ | [m] | Radius |
| $Re$ | [-] | Reynolds number |
| $r_{(0)}$ | [m] | Distance of point $x'_k$ from vortex sheet |
| $v_{(av)}$ | [m.s$^{-1}$] | Average velocity between two parallel plates |
| $v_{(x)}$ | [m.s$^{-1}$] | Component of velocity in x direction |
| $v_{(x\ max)}$ | [m.s$^{-1}$] | Maximal velocity component in x direction |
| $v^*$ | [m.s$^{-1}$] | Shear velocity |
| $v^+$ | [-] | Dimensionless velocity near the wall |
| $v_i$ | [m.s$^{-1}$] | Velocity vector |
| $v_2, v_2, v_3$ | [m.s$^{-1}$] | Components of velocity vector |
| $x_1, x_2, x_3$ | [m] | Coordinates of location |
| $x'_k$ | [m] | Coordinates of the induced velocity point |
| $x_{(0)k}$ | [m] | Coordinates of point $x'_k$ projected onto the vortex sheet |
| $y$ | [m] | Coordinate y, or distance from wall |
| $y^+$ | [-] | Dimensionless distance from the wall |
| $\Gamma$ | [m$^2$.s$^{-1}$] | Circulation |
| $\gamma$ | [m.s$^{-1}$]/ [s$^{-1}$] | Linear /planar vorticity density |
| $\mu$ | [Pa.s] | Dynamic viscosity |
| $\nu$ | [m$^2$.s$^{-1}$] | Kinematic viscosity |
| $\varepsilon_{ijk}$ | [-] | Levi-Civita tensor |
| $\tau_\mu$ | [Pa] | Viscous shear stress |
| $\tau_t$ | [Pa] | Turbulent (Reynolds) shear stress |

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**References**

Pope, S.B. (2008) Turbulent Flows. Fifth printing, Cambridge University Press, ISBN 978-0-521-59889-6.

Janalík, J. (2008) Vybrané kapitoly z mechaniky tekutin. VŠB – Technická univerzita Ostrava, ISBN 978-80-248-1910-5

Munson, B.R. & Young, D. F. & Okiishi, T.H. (2006) Fundamentals of Fluid Mechanics, Fifth Edition, John Wiley & Sons, Inc., ISBN 978-0-471-67582-2.

Brdička, M. & Samek, L. & Sopko, B. (2000) *Mechanika kontinua*. Akademie věd České republiky, ACADEMIA, ISBN 80-200-0772-5.

Lewis, R.I. (1991) *Vortex Element Mehods for Fluid Dynamics Analysis of Engineering Systems.* Cambridge Engine Technology Series, Cambridge University Press, ISBN 0-521-36010-2.