18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012

pp. 1161–1171
Paper #122

MASS LUMPING METHODS FOR THE SEMI-LOOF SHELL ELEMENT

Vít Sháněl [*], **Radek Kolman** [**] **and Jiří Plešek** [***]

Abstract: *Mass matrix diagonalization in terms of a finite element method (FEM) is essential for an effective deployment of the explicit method as one of the direct integration methods of the motion equations of elastodynamics. A particular attention is focused on the mass matrix diagonalization of the semi-loof shell element. Its diagonalization requires a specially designed universal diagonalization scheme that is derived from the scaling HRZ method. Another analyzed aspect is the problem of preserving the moment of inertia for various types of finite elements. The proposed scheme is implemented in the finite element program and consequently tested on several problems.*

Keywords: ***FEM, mass matrix lumping, semi-loof finite shell element, impacts and waves in solids, elastodynamics.***

1. Introduction

The most suitable approach to numerically solving problems of elastodynamics turns out to be a finite element method (FEM) [Hughes (1987)] that seems to provide the most effective way of dealing with practical cases. A vast majority of those cannot be worked out using analytical closed-form solutions, or the solutions are just too difficult. FEM is a wonderfully versatile method that can be used even in the most complex constitutive equations and which enables the approximation of a wide range of boundary conditions [Zienkiewicz et al. (2005)]. After applying the FEM to elastodynamics problems you will obtain

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}_{ext} \tag{1}$$

where $\mathbf{q}$ is a nodal displacement, $\ddot{\mathbf{q}}$ nodal acceleration, $\mathbf{F}_{ext}$ represents external forces, $\mathbf{K}$ denotes the stiffness matrix, which is symmetric, and after applying boundaries conditions also positive semidefinite and $\mathbf{M}$ is a symmetric and positive definite mass matrix given by the integral

$$\mathbf{M} = \int_V \rho \mathbf{H}^T \mathbf{H} dV = [m_{ij}] \tag{2}$$

where ρ is mass density, $\mathbf{H}$ matrix of shape functions and V represents elements volume.

There is a large number of different numerical methods that are related to a FEM-based spatial discretization [Zienkiewicz et al. (2005)]. One of the most frequently implemented methods is a so called semidiscretization which initially performs a spatial discretization and then runs an independent time discretization. The time part of any problem is usually dealt with using methods of direct integration, which can be implicit or explicit [Hughes (1987)]. This work focuses especially on explicit methods, notably the central difference method [Dokainish (1989)] which deals with time derivations this way

$$\dot{\mathbf{q}}^t = \frac{1}{2\Delta t}\left(\mathbf{q}^{t+\Delta t} - \mathbf{q}^{t-\Delta t}\right) \quad \text{a} \quad \ddot{\mathbf{q}}^t = \frac{1}{\Delta t^2}\left(\mathbf{q}^{t+\Delta t} - 2\mathbf{q}^t + \mathbf{q}^{t-\Delta t}\right) \tag{3}$$

[*]Ing. Vít Sháněl: Institute of Thermomechanics AS CR, Impact and Waves in Solids, Dolejškova 1402/5, Prague, CZ, email: shanel@it.cas.cz
[**]Ing. Radek Kolman, Ph.D.: Institute of Thermomechanics AS CR, Impact and Waves in Solids, Dolejškova 1402/5, Prague, CZ, email: kolman@it.cas.cz
[***]Ing. Jiří Plešek, CSc.: Institute of Thermomechanics AS CR, Impact and Waves in Solids, Dolejškova 1402/5, Prague, CZ, email: plesek@it.cas.cz

where Δt denotes time step and $\mathbf{q}^{t-\Delta t}$, $\mathbf{q}^{t}$ and $\mathbf{q}^{t+\Delta t}$ are nodal displacements in previous, actual and next time step, respectively. In order to ensure effectivity of this method, explicit formulation of $\mathbf{q}^{t+\Delta t}$, it is vital to stress the importance of the mass matrix of the system which needs to be diagonal – as you can see in equation (4) which combines (2) and (3)

$$\frac{1}{\Delta t^2}\mathbf{M}\mathbf{q}^{t+\Delta t} = \mathbf{F}_{ext} - \left(\mathbf{K} - 2\mathbf{M}/\Delta t^2\right)\mathbf{q}^{t} - \left(\mathbf{M}/\Delta t^2\right)\mathbf{q}^{t-\Delta t} \tag{4}$$

Only then, with diagonal mass matrix $\mathbf{M}$, becomes the related inversion simple and the actual computation is elementary. However, a consistent mass matrix does not fulfill this requirement generally. Hence a great effort has been made to develop some diagonalization methods [Zienkiewicz et al. (2005)].

The mass matrix diagonalization may involve some changes in fundamental physical properties (e.g. mass, the moment of inertia) which could have a significant impact on the system behavior. It is therefore desirable to devise a method that preserves those system parameters to the highest possible degree and at the same time allows for a universal application on a wide range of finite elements.

1.1. Semi-loof Shell elements

A versatility of the isoparametric elements has been transferred to a semi-loof shell element during the years of the research conducted by Irons and his team, related to the study of semi-loof shell element [Hellen (1986)] such as sharp edges, curved sides and surfaces and multiply-connected regions. This element has been widely used for elastic analysis and later extended for material and geometric nonlinearities. The element is available in the 8-node quadrilateral and 6-node triangle form which significantly aids the meshing process. The topology of the semi-loof element is depicted in Figure 1 where

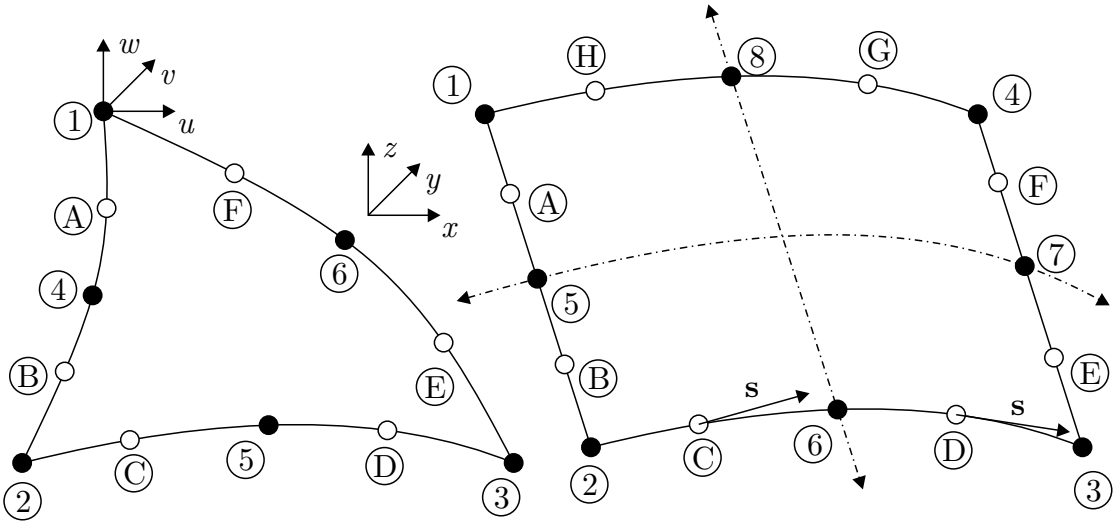

Fig. 1: Semi-loof element configuration [Irons (1976)]

1 to 6 or 8 (represented as a filled circle) are the classic nodes containing three translational DOFs u, v and w. The local normal rotation $\mathbf{s}$ acts at each loof node labeled A to F or H (empty circle). They are situated at side position ± 0.57735 from the midside nodes outwards. The normal is defined in each loof node, but they are identified to the adjacent midside node and therefore a user does not have to define a loof node geometry. Thus nodes A and B are coupled with the midside node 4 (or 5 in a quadrilateral geometry) so that the DOFs in midside node become u, v, w, $\mathbf{s}_1$ and $\mathbf{s}_2$. The element employs three different shape functions [Leština (1980)]. A membrane deformation is described using isoparametric biquadratic shape functions and a bending deformation is approximated by loof and bubble functions.

2. Diagonalization method analysis and design

2.1. HRZ scheme

Also called diagonal scaling method, the essential idea of which is contained in [Hinton et al. (1976)] where Hinton, Rock and Zienkiewicz says: *"The procedure of lumping recommended in view of the infinite possibilities offered by condition* (5) *is to compute the diagonal terms of the consistent mass matrix and then scale these terms so as to preserve the total mass of the element."*

$$\begin{aligned} \bar{H}_i &= 1 \quad \text{at} \quad i\text{th node} \\ \bar{H}_i &= 0 \quad \text{at} \quad j\text{th node} \\ \sum \bar{H}_i &= 1 \end{aligned} \tag{5}$$

This scheme has a plenty of advantages compared to other lumping techniques. Mainly, the HRZ scheme is able to diagonalize mass matrices of the elements which include rotational as well as translational DOFs. A modification of the procedure which ensures rotational DOFs scaling is apparent from the algorithm description below. Another advantage of this scheme is that an assemblage of the whole consistent matrix is not necessary – only diagonal terms are required and the implementation is performed on an element level.

The HRZ procedure is as follows [Fellipa (2011)]:

1. Compute diagonal terms of the consistent mass matrix $\mathbf{M}_c$.
2. Determine the element mass m^e.
3. For each coordinate direction, select the DOFs – translational and rotational – which contribute to the motion in the respective direction.
4. Segment these selections into sub-selections of the rotational and translational DOFs.
5. Sum all the terms of the each translational sub-selections, call this sum S_i, where i is the problem dimension.
6. Multiply all the entries in the *i*th selection (step 3) by the m^e/S_i for all i coordinate directions.

2.2. Conservation of the moment of inertia

The HRZ lumping method conserves only the mass of the element but no moment of inertia [Fellipa (2011)]. This was a motivation for the modification of the HRZ algorithm to conserve the element rotational inertia as well.

The algorithm is based on the idea to calculate three rotational kinetic energies for the consistent mass matrix induced by a set of three orthogonal angular velocity vectors (Figure 2). What follows is the execution of the HRZ lumping process on the translational DOFs and a calculation of the kinetic energy induced by the same group of angular velocities. In the last step the angular kinetic energies are compared and correction coefficients for rotational DOFs are computed in order to preserve the total moment of inertia.

The basic input for the created algorithm is a consistent mass matrix of the element, coordinates of the nodes and also the information about DOFs in every node. In this example vectors describing translational and rotational DOF position were assembled manually. A definition of the three orthogonal angular velocities uses a Miller indices notation which describes a turn of the coordinate system of the three velocities. In general, the velocity orientation can be chosen arbitrarily but it is very useful for the algorithm testing of this approach which allows an easy change in orientation of the input velocities.

All the computations below are performed for all of the three input angular velocities in a vector $\boldsymbol{\omega}_i$ where $i = 1, 2, 3$ which are denoted in the source code with indices x, y and z.

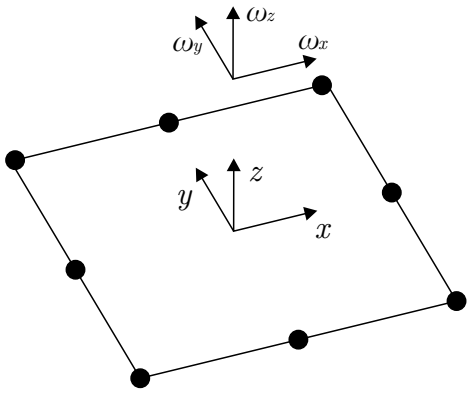

Fig. 2: Orientation of the three angular velocity vectors

Translational degrees of freedom

So as to translate the DOF scaling, it is necessary to have a consistent mass matrix. In result, the scaling coefficient α_{ti} for the translational DOFs is computed

$$\alpha_{ti} = \frac{\mathbf{v}_{t_i}^T \, \mathbf{M}_c \, \mathbf{v}_{ti}}{\mathbf{v}_{t_i}^T \, \text{diag}\, \mathbf{M}_c \, \mathbf{v}_{ti}} \tag{6}$$

where $\mathbf{v}_{ti}$ consists of the nodal velocities corresponding to the translation rigid body motion in directions x, y and z. Then α_{ti} is used for the scaling of the terms on the $\mathbf{M}_c$ diagonal to preserve the element mass.

$$\mathbf{M}_t^l = \boldsymbol{\alpha}_t \mathbf{P}_t \, \text{diag}\, \mathbf{M}_c \tag{7}$$

where $\mathbf{P}_t$ is a distribution matrix of the components of $\boldsymbol{\alpha}_t = (\alpha_{ti})$ to the corresponding terms in $\mathbf{M}_c$. This scaling method is an equivalent to the classic HRZ.

Rotational degrees of freedom

In order to compute rotational kinetic energy the projection of the angular velocity vectors to the nodes is necessary, thus the peripheral velocity in the nodes is a result of the vector product

$$\mathbf{v}_{\mathbf{p}ij} = \boldsymbol{\omega}_i \times \mathbf{p}_j \tag{8}$$

where $\mathbf{v}_{\mathbf{p}_{ij}}$ is peripheral velocity caused by one of the three induced angular velocities $\boldsymbol{\omega}_i$ and the position coordinates of the *j*th node in the global coordinate system $\mathbf{p}_j$. In that case, the velocities $\mathbf{v}_{p_{ij}}$ are arranged in the vector $\mathbf{v}_{g_i}$ which corresponds to the positions of the DOFs in the mass matrix.

The scalar product of the input angular velocities with the directions of the rotational DOFs is used here as an angular velocity projection to the rotational DOFs

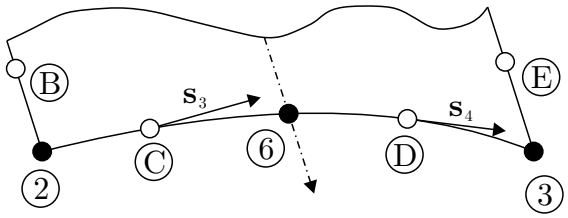

Fig. 3: The orientation of the rotational DOFs in the semi-loof shell element

$$\boldsymbol{\omega}_{p_{ij}} = (\boldsymbol{\omega}_i \cdot \mathbf{s}_j) \cdot \mathbf{s}_j \tag{9}$$

where $\mathbf{s}_j$ is the directional unit vector of the *j*th rotational loof node (Figure 3) oriented from the node with a lower number to the node with a higher one and they are tangential to the element border. Then the

Tab. 1: Conservation of the moment of inertia

	Kinetic energy [J]		
	$E_{k\,c}$ (consistent)	$E_{k\,l}$ (lumped)	$E_{k\,lm}$ (lumped + modified)
x	3.25416666×10^{-3}	5.55154715×10^{-3}	3.67418730×10^{-3}
y	3.25416666×10^{-3}	5.55154715×10^{-3}	3.67418730×10^{-3}
z	3.11666666×10^{-3}	4.98978214×10^{-3}	2.48663570×10^{-3}

Tab. 2: Conservation of the moment of inertia – error %

	Error of the kinetic energy [%]	
ε_r	lumped	lumped + modified
x	70.6	12.9
y	70.6	12.9
z	60.1	-20.2

vector $\mathbf{v}_g$ of all (translational and rotational) projected velocities (peripheral and angular) is assembled and the total kinetic energy $E_{k\,c}$ of the consistent mass matrix can be computed as

$$E_{k\,ci} = \frac{1}{2}\mathbf{v}_{g_i}^T\mathbf{M}_c\mathbf{v}_{g_i} \tag{10}$$

and equally for lumped mass matrix

$$E_{k\,li} = \frac{1}{2}\mathbf{v}_{g_i}^T\mathbf{M}_l\mathbf{v}_{g_i} \tag{11}$$

where the lumped mass matrix consist of two mass matrices

$$\mathbf{M}_l = \mathbf{M}_t^l + \mathbf{M}_r^l \tag{12}$$

The first one contains only terms corresponding to the translational DOFs where $\boldsymbol{\alpha}_t$ is computed in equation (6) and the second one $\mathbf{M}_r^l$ contains only terms corresponding to the rotational DOFs

$$\mathbf{M}_r^l = \boldsymbol{\alpha}_r\mathbf{P}_r\mathrm{diag}\mathbf{M}_c \tag{13}$$

where $\mathbf{P}_r$ is a distribution matrix of the components of $\boldsymbol{\alpha}_r = (\alpha_{ri})$ to the corresponding terms in $\mathbf{M}_c$. To conserve kinetic energy of the element during the lumping process, the energies in (10) and (11) must be equal $\mathbf{E}_{k\,c} = \mathbf{E}_{k\,l}$. The substitution of the equations (10) and (11) will result in

$$\mathbf{v}_{g_i}^T\mathbf{M}_c\mathbf{v}_{g_i} = \mathbf{v}_{g_i}^T\mathbf{M}_l\mathbf{v}_{g_i} \tag{14}$$

and after another substitution with equation (12), (7) and (13) we will get

$$\mathbf{v}_{g_i}^T\mathbf{M}_c\mathbf{v}_{g_i} = \mathbf{v}_{g_i}^T\boldsymbol{\alpha}_t\mathbf{P}_t\mathrm{diag}\mathbf{M}_c\mathbf{v}_{g_i} + \mathbf{v}_{g_i}^T\boldsymbol{\alpha}_r\mathbf{P}_r\mathrm{diag}\mathbf{M}_c\mathbf{v}_{g_i} \tag{15}$$

Equation (15) consist of three equations ($i = 1, 2, 3$) and only the vector $\boldsymbol{\alpha}_r$ is an unknown. In this case $\boldsymbol{\alpha}_r$ contains eight generally different terms and (15) just three equations which can be rewritten as

$$\mathbf{A}\boldsymbol{\alpha}_r = \mathbf{E}_{k\,c} - \mathbf{E}_{k\,t} \tag{16}$$

and for $\boldsymbol{\alpha}_r$

$$\boldsymbol{\alpha}_r = \mathbf{A}^+\left(\mathbf{E}_{k\,c} - \mathbf{E}_{k\,t}\right) \tag{17}$$

where $\mathbf{A}^+$ denotes Moore-Penrose pseudo-inverse. Then the lumped mass matrix can be easily assembled according to the equation (12).

The system of equations (17) is over-determined, therefore pseudo-inversion has to be employed. Then the solution of this system is accordance with the solution using the least squares method. This method does not provide an exact solution which is impossible due to the system over-determination but the final solution is an approximation as can be seen in Table 1. The final kinetic energies will not be exactly the same as the ones computed from the CMM. In Table 2 there is a relative error in percentage of the lumped and lumped + modified kinetic energy where ε_r is for kinetic energy of the lumped mass matrix computed as

$$\varepsilon_r = 100 \cdot \frac{E_{k\,c} - E_{k\,l}}{E_{k\,c}} \quad \text{and for lumped + modified} \quad \varepsilon_r = 100 \cdot \frac{E_{k\,c} - E_{k\,lm}}{E_{k\,c}} \tag{18}$$

It is clear that the first two kinetic energies computed from the modified DLMM are slightly larger then the ones from CMM but the third one is smaller – precisely in compliance with the approximation assumption mentioned above.

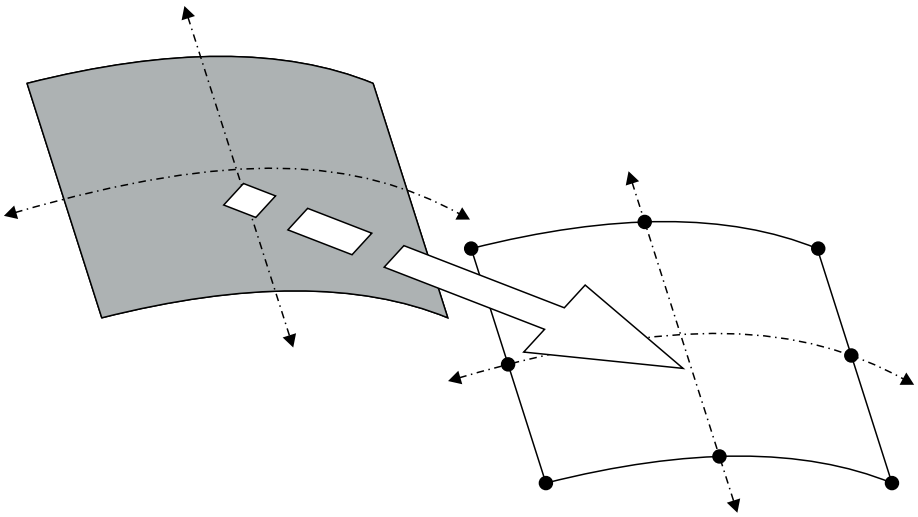

Fig. 4: The mass lumping process

The algorithm described above works well. The angular kinetic energy of the lumped and modified mass matrix is preserved – considering the feasibility of the last square method. On the other hand negative entries in the lumped mass matrix are detected. The reason for that is obvious provided that the lumping process has been performed and the element mass has been concentrated to the nodes. If the element mass is preserved and stored to the most outlying places – the nodes, the element moment of inertia has to increase together with the rotational kinetic energy when the angular velocity is introduced. In order to preserve the rotational kinetic energy, the rotational DOFs contained in the semi-loof element have to be negative to suppress the effect of the lumping process to the moment of inertia.

2.3. Rotational degrees of freedom

The fundamental question in mass matrix lumping is the importance of the rotational DOFs. How significant is their contribution to the kinetic energy when we work with a semi-loof shell element? In the previous section we investigated the rotational DOF contribution in order to produce a lumped mass matrix with the equal inertia characteristic and also described the production of negative entries. In the Section 2.1. we explained the HRZ algorithm with the modification to deal with rotational DOFs. To use this algorithm for a diagonalization semi-loof element, it is necessary to decide in which direction they contribute to the translational motion. It is not easy and clear-cut to determine, and when this decision is made the rotational DOFs are scaled without any physical reason. Therefore, let's examine and focus on the importance and the contribution of the rotational DOFs of the semi-loof element on the influence of the rotational kinetic energy during the diagonalization process.

The rotational DOF influence was tested on the plate (Figure 5) of the dimensions width w = 1 mm, height h = 1 mm, thickness t_1 = 0.1 mm and t_2 = 0.01 mm, mass density $\rho = 7800\,\mathrm{kg/m^3}$ and the angular

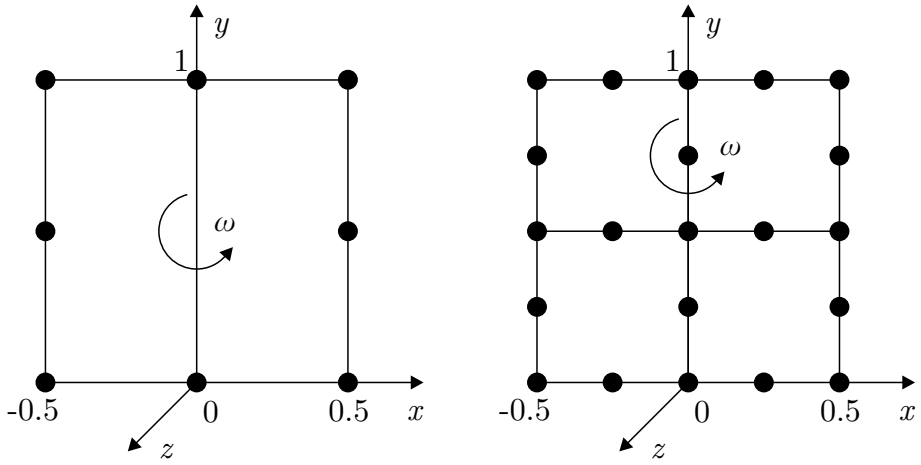

Fig. 5: Influence of the rotational DOFs ($n_{el} = 1$ and 2)

Tab. 3: Influence of the rotational DOF – semi-loof element, thickness = 0.1 mm

	Angular kinetic energy [J]	
n_{el}	with rotational DOFs	without rotational DOFs
1	$5.936942830255\times10^{-8}$	$5.93694266761\times10^{-8}$
2	$3.926416299426\times10^{-8}$	$3.926416251202\times10^{-8}$
4	$3.42309784091590\times10^{-8}$	$3.42309782178395\times10^{-8}$
8	$3.29567308352155\times10^{-8}$	$3.29567307279155\times10^{-8}$

velocity ω = 1 rad/s. This plate was discretized into four meshes differing in density. All four discretized plates are displayed in Figures 5 and 6 where n_{el} denotes the number of elements along the plate side. In Table 3 you can see the kinetic energy of all four plates with and without considering the rotational DOFs obtained after applying angular velocity ω for thickness t_1 = 0.1 mm and in Table 4 for element thickness t_1 = 0.01 mm.

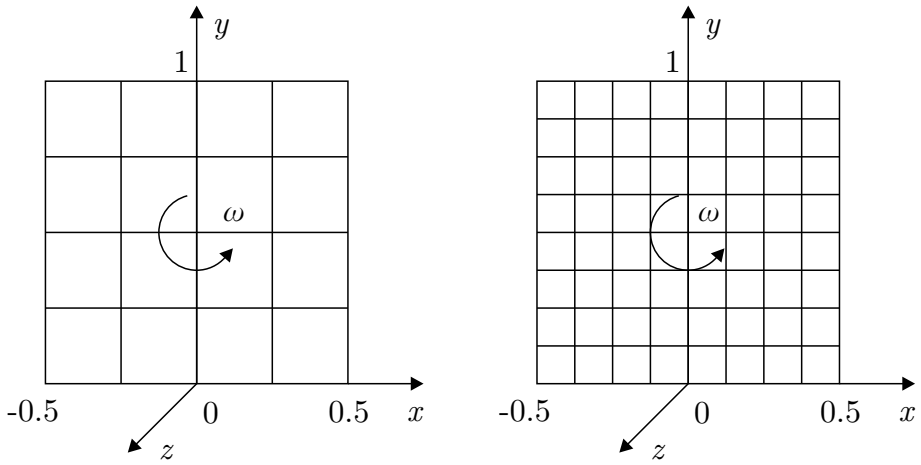

Fig. 6: Influence of the rotational DOFs ($n_{el} = 4$ and 8)

It is important to realize that whether we keep the original entries on the mass matrix diagonal corresponding to the rotational DOFs or reset the values to zero, the differences in results are almost identical. Further information about the influence of the rotational DOFs on a semi-loof element can be

Tab. 4: Influence of the rotational DOF – semi-loof element, thickness = 0.01 mm

	Angular kinetic energy [J]	
n_{el}	with rotational DOFs	without rotational DOFs
1	$5.930478568835\times10^{-9}$	$5.930478416375\times10^{-9}$
2	$3.920169155838\times10^{-9}$	$3.9201691176445\times10^{-9}$
4	$3.4175917166041\times10^{-9}$	$3.4175917069787\times10^{-9}$
8	$3.29194701535405\times10^{-9}$	$3.29194701287015\times10^{-9}$

found in Figure 7 where the error is computed as

$$\varepsilon_r = 100 \cdot \frac{|E_{k\,l} - E_{k\,lr}|}{E_{k\,l}} \tag{19}$$

where $E_{k\,l}$ is the kinetic energy of the plate with a lumped mass matrix while considering the rotational DOFs, and $E_{k\,lr}$ is the same energy without considering the rotational DOFs.

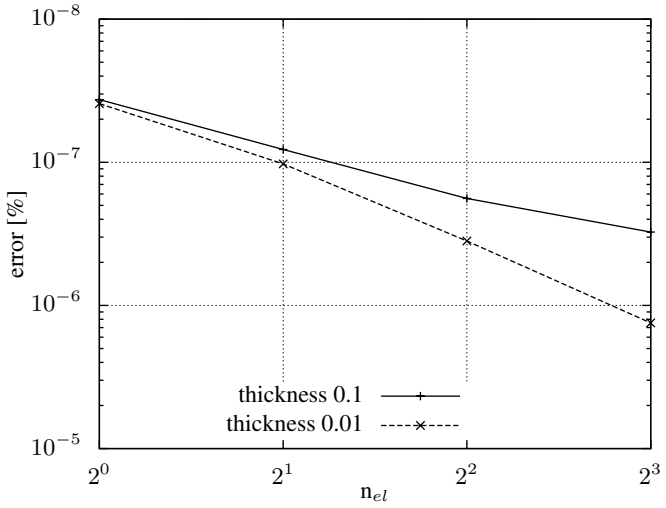

Fig. 7: The influence of rotational DOF on kinetic energy

The influence of the rotational DOFs on the kinetic energy of the rotating plate is almost negligible. If the plate consists of only one semi-loof element the contribution of the rotational DOFs to the kinetic energy is approximately 2.7×10^{-6} %, and the denser the mesh the smaller the contribution.

Having established these facts, we can now omit the rotational DOFs in the diagonalization process and keep only the original entries which come out from the consistent mass matrix, corresponding to the variational principles as only these values have a physical reason.

3. Results and Discussion

The previous sections of this paper dealt with the difference between consistent and lumped mass matrices and described the influence of this difference on the kinetic energy. It is now highly important to prove that the kinetic energy of the plate (Figures 5 and 6) converges to the analytical solution when the mesh contains more elements.

A technique which we already used for testing an algorithm convergence is similar to the one used in Section 2.3. to check the influence of the rotational DOFs on the kinetic energy – the same testing plates

with the induced angular velocity ω will also be used and those can be chosen arbitrarily. The angular velocity ω was projected to the loof nodes direction and the result was reassembled into vector $\mathbf{v}_g$ as described in Section 2.2.. The kinetic energy was calculated as

$$E_{k\,c} = \frac{1}{2}\mathbf{v}_g^T \mathbf{M}_c \mathbf{v}_g \tag{20}$$

Figure 9 shows the kinetic energy convergence of the numerical solution using the consistent mass matrix as well as the analytical one with the raising ratio of element width to element thickness. The error is computed as

$$\varepsilon_r = 100 \cdot \frac{|E_k - E_{k\,c}|}{E_k} \tag{21}$$

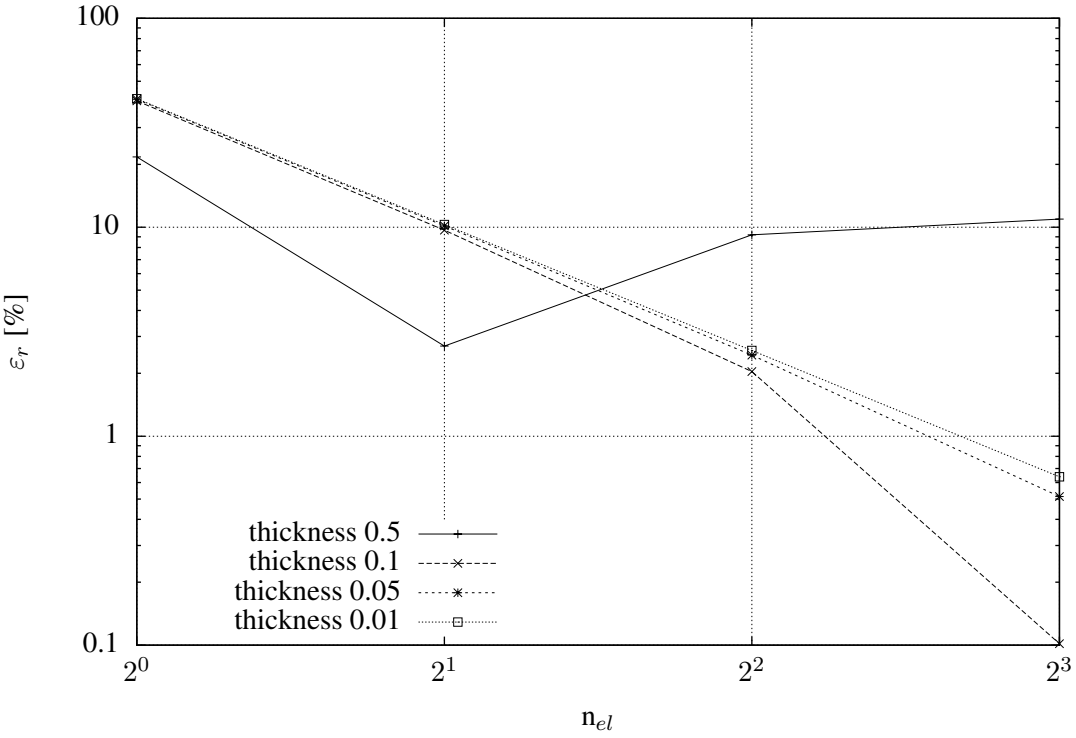

Fig. 8: Relative error in the kinetic energy corresponding to the numerical solution with CMM and DLMM – convergence test

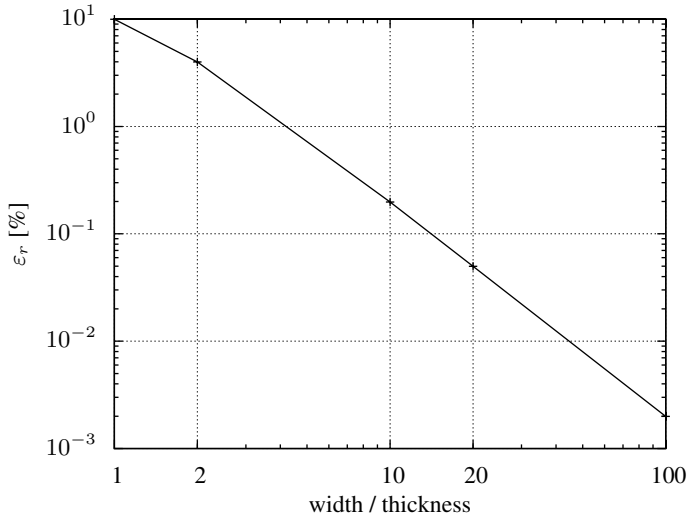

Fig. 9: Relative error in the kinetic energy corresponding to the analytical and numerical solution (CMM) – convergence test

where E_k is the analytical, exact, solution and $E_{k\,c}$ the numerical one with a consistent mass matrix. This test proved that the solution for kinetic energy of the plate modelled with semi-loof shell element converges to the analytical solution.

Figures 8 and 10 show a convergence rate of the solution with the lumped mass matrix to the solution with the consistent mass matrix (Figure 8) and to the analytical solution (Figure 10).

4. Conclusions

Due to the nature of the requirements related to the process of diagonalization (a universal algorithm, preserving the mass, an easy implementation), a method called HRZ was ultimately selected as a frequently-used approach that was the most suitable option for a consequent implementation. A special attention was paid to mass matrices of elements that contained rotational degrees of freedom – such as shells. Therefore a modification of this scaling method had to be devised not only in order to preserve the element mass, but also to preserve the moment of inertia. This method unfortunately returned negative terms in mass matrices. They lost their positive definiteness and consequently also became useless for an explicit solver implemented in the PMD program.

The standard HRZ method implementation involves a scaling of the rotational degrees of freedom in the same way which applies to the translational degrees of freedom, and it has no physical basis, thus it does not provide any solution for semi-loof shell elements in this form. This was the reason why the interest shifted entirely on the problem of a mass matrix diagonalization with rotational degrees of freedom in which we determined the influence of the rotational degrees of freedom on the total moment of inertia of the semi-loof element. The numerical test proved that the influence of the rotational degrees of freedom was negligible. And hence the implemented algorithm was designed to leave the original values belonging to the rotational degrees of freedom on the diagonal without any changes whatsoever.

Convergence tests were then carried out to observe the kinetic energy of the rotating element which would confirm a convergence to the consistent matrix as well as, most importantly, a convergence to the analytically calculated values. The implemented algorithm was tested on problems which involved natural frequencies and the shock wave propagation.

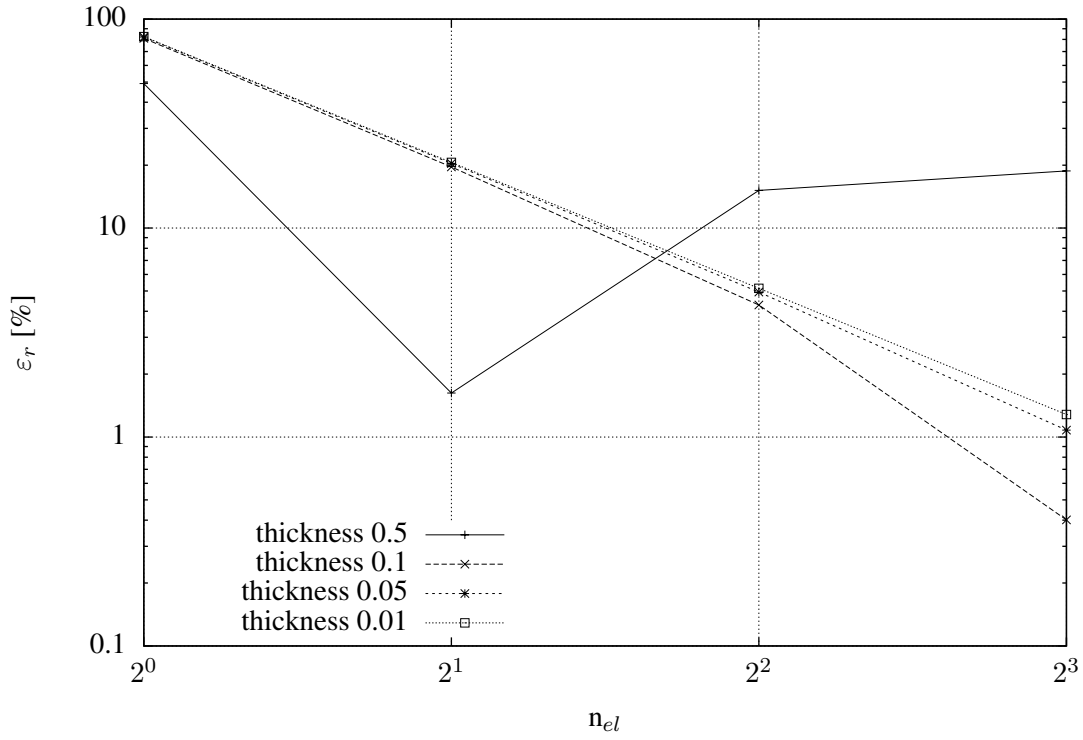

Fig. 10: Relative error in the kinetic energy corresponding to the analytical and numerical solution (DLMM) – convergence test

Acknowledgments

This work was supported by the grant projects GA CR 101/09/1630, ME10114, P101/10/P376 and P101/11/0288 under AV0Z20760514.

References

Cook, R. D., Malkus, D. S. & Plesha, M. E. (2002) *Concepts and applications of finite element analysis.* Wiley, New York.

Dokainish, M. A. & Subbaraj, K. (1989) A survey of direct time-integration methods in computational structural dynamics—I. explicit methods. *Computers & Structures*, 32, 6, pp. 1371–1386.

Fellipa, C. (2011) *Introduction to finite element methods (asen 5007) – fall 2010*, online (March 11, 2011). *http://www.colorado.edu/engineering/cas/courses.d/IFEM.d/*

Hellen, T. K. (1986) An assessment of the semiloof shell element, *International Journal for Numerical Methods in Engineering*, 22, 1, pp. 133–151.

Hinton, E., Rock, T. & Zienkiewicz, O. (1976) A note on mass lumping and related processes in the Finite element method. *Earthquake Engineering and Structural Dynamics*, 13, 9, p. A112.

Hughes, T. J. R. (1987) *The Finite Element Method*, Prentice-Hall Inc., New Jersey.

Irons, B. (1976) *Finite Elements for Thin Shells and Curved Membranes*, Wiley, London, Ch. The semi-loof shell elements.

Leština, J. (1980) Tenkoskořepinový konečný prvek semiloof, *Strojírenství*, 39, 12, pp. 714–719.

Sháněl, V. (2011) *On the Mass Lumping in the Finite Element Method.* Master Thesis, CTU in Prague, Faculty of Mechanical Engineering, Prague.

Zienkiewicz, O., Taylor, R. & Zhu, J. (2005) *Finite Element Method*, sixth Edition, Elsevier, Oxford.