18th International Conference
ENGINEERING MECHANICS 2012
Svratka, Czech Republic, May 14 – 17, 2012

pp. 1145–1156
Paper #45

LOCAL INTEGRAL FORMULATIONS FOR THIN PLATE BENDING PROBLEMS

L. Sator, V. Sladek, J. Sladek*

Abstract: *In this paper, we present that the decomposition of the biharmonic equation into two Poisson equations is applicable to general case of boundary conditions and any shape of the boundary edge of the plate, if we use the Local Integral Equation (LIE) formulation and a meshless approximation for primary field variables. Besides the standard advantages of mesh free formulations remember the new advantage consisting in decreasing the order of the derivatives of field variables. Instead of the third order derivatives of the deflection field in the weak formulation for the biharmonic equation the highest order of the derivatives in the present weak formulation does not exceed the first order. Mostly, it is decreased also the order of the derivativesof new field variables in the expressions of the boundary conditions. Several illustrative examples are presented for comparison of accuracy, convergence and computational efficiency achieved by using various approaches.*

Keywords: ***Local Integral Equation formulation, meshless approximation, decomposition, biharmonic equation, Poisson equations***

1. Introduction

It is well known that high order derivatives of field variables in the governing equations give rise to difficulties in solution of boundary value problems because of worse accuracy of numerically evaluated high order derivatives. The order of the differential operator can be decreased by decomposing this operator into two lower order differential operators with introducing new field variables. The relevant boundary densities in the decomposed problem are different from the boundary densities in the original boundary value problem. Therefore, sometime it can be problematic to express the original boundary conditions in terms of the new field variables and/or their derivatives. Especially, it is impossible in general in the boundary element formulations where the unknowns are localized and approximated on boundary alone.

In two recent decades, solution of many engineering problems as well as problems of mathematical physics have been reformulated by using various mesh free formulations with meshless approximations. Such approximations belong to domain type approximations and the restrictions of boundary elements can be eliminated.

In this paper, we present that the decomposition of the biharmonic equation into two Poisson equations is applicable to general case of boundary conditions and any shape of the boundary edge of the plate, if we use the weak formulation based on the Local Integral Equations (LIE) formulation and a meshless approximation for primary field variables. In the local weak formulation, the constant test function with the support on the sub-domain is utilized, what corresponds to integral satisfaction of physical balance principles (equilibrium of forces and force moments) on local sub-domains. To illustrate the robustness of the proposed formulation, we present also the weak formulation for the original biharmonic problem. The strong formulation is not considered because of the 4^{th} order derivatives of deflections. Owing to the decomposition, the order of the derivatives in the governing equations is decreased form four to two. The prescribed boundary conditions are considered in each formulation in

* Ing. Ladislav Sátor, Prof. RNDr. Vladimír Sládek, DrSc., Prof. Ing. Ján Sládek, DrSc., Institute of Construction and Architecture, Slovak Academy of Science, 845 03 Bratislava, Slovakia, e-mail: ladislav.sator@savba.sk

strong form by collocation at boundary nodes. Two kinds of the meshless approximations are employed in this paper, such as the Moving Least Square (MLS) approximation (Lancaster and Salkauskas, 1981) and the Point Interpolation Method (PIM) (Liu, 2003). Besides the standard advantages of mesh free formulations remember that the domain-type character of the approximation enables us to express all the boundary quantities in terms of the new field variables and/or their derivatives on the boundary. Numerical examples with exact benchmark solution are considered for comparisons of accuracy, convergence and computational efficiency of various approaches.

2. Physical decomposition of the governing equations

In Kirchoff's theory of bending of thin plates (Timoshenko and Woinowsky-Krieger, 1959) the all physical quantities are expressed in terms us the functions of deflection $\boldsymbol{w(x)}$ and/or their derivatives. For the plate of thickness h and midplane Ω orthogonal to the axis x_3, the tensor of moments can be expressed in terms of the second order derivative of deflection as

$$M_{ij} = -D\left[(1-\nu)w_{,ij} + \nu\delta_{ij}\nabla^2 w\right], \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{1}$$

where D is the bending stiffness, E and ν is the Young modulus and Poisson ratio, respectively.
The bending moment M and the twisting moment T on the boundary edge $\Gamma = \partial\Omega$ are given as

$$M = n_i n_j M_{ij} = -D\left[(1-\nu)n_i n_j w_{,ij} + \nu\nabla^2 w\right] \tag{2}$$

$$T = n_i t_j M_{ij} = -D(1-\nu)n_i t_j w_{,ij} = -D(1-\nu)t_i n_j w_{,ij} \tag{3}$$

where n_i and t_i are the Cartesian components of the unit normal and tangent vector on Γ, respectively. The transversal the shear force N and the equivalent shear force on the boundary edge are defined as

$$N := n_i M_{ij,j} = -n_i D\nabla^2 w_{,i}, \qquad V := N + \frac{\partial T}{\partial \mathbf{t}} \tag{4}$$

The governing equation for deflections of thin plane is given as

$$M_{ij,ij}(\mathbf{x}) = -q(\mathbf{x}), \tag{5}$$

hence after substituting (1_1) to (5) we can obtain governing equation in form

$$\left[D(1-\nu)w_{,ij}\right]_{,ij} + \nabla^2\left[\nu D\nabla^2 w\right] = q \tag{6}$$

with $q(\mathbf{x})$ being the density of transversal loading applied on the plate surface.

Usually, Poisson ratio is constant and the governing equation becomes

$$D\nabla^2\nabla^2 w + 2D_{,i}(\nabla^2 w)_{,i} + (1-\nu)D_{,ij}w_{,ij} + \nu(\nabla^2 D)(\nabla^2 w) = q\,. \tag{7}$$

If we shall consider the bending stiffness to be constant, then the governing equation is simplified as

$$D\nabla^2\nabla^2 w = q\,. \tag{8}$$

Three basic boundary conditions can be assumed on the boundary edge Γ:

(i) clamped edge: $w\big|_\Gamma = 0;\ \left.\frac{\partial w}{\partial n}\right|_\Gamma = 0$

(ii) simply supported edge: $w\big|_\Gamma = 0;\ M\big|_\Gamma = 0$ (9)

(iii) free edge: $M|_\Gamma = 0$; $V|_\Gamma = 0$

The fourth order derivatives of deflections in governing equations can give rise to serious difficulties not only in strong formulation for numerical solution, but also in weak formulation owing to inaccurate approximation of high order derivatives of deflections occurring in the integral equations as well as in boundary conditions.

Therefore, it is expedient to introduce the new field variable defined as

$$m(\mathbf{x}) := -D\nabla^2 w(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\Omega \tag{10}$$

Then the governing equation (8) is split into two equations given by (10) and (11)

$$\nabla^2 m(\mathbf{x}) = q(\mathbf{x}) \quad \text{for..} \quad \mathbf{x}\in\Omega \tag{11}$$

3. Weak formulation of governing equations

The weak formulations of governing equations (10) and (11) corresponding to two field variables with assuming the bending stiffness to be constant, can be written as

$$D\int_{\partial\Omega^s} n_i w_{,i} d\Gamma + \int_{\Omega^s} m d\Omega = 0 \tag{12}$$

$$\int_{\partial\Omega^s} n_i m_{,i} d\Gamma = -\int_{\Omega^s} q d\Omega \tag{13}$$

The relevant boundary quantities $\{w, \partial w/\partial\mathbf{n}, M, V\}$ are expressed in terms of two field variables and their derivatives on boundary edges.

4. Meshless approximation of fields variables

In general, a meshless approximation uses a local interpolation to represent trial function with the values of the unknown variables at some randomly distributed nodes. Now, we shortly describe two kinds of meshless approximations. For the sake of brevity, we shall use the common notation $u(\mathbf{x})$ for the scalar fields $w(\mathbf{x})$ and/or $m(\mathbf{x})$.

Moving Least Square (MLS) approximation

In the MLS-approximation, the polynomial basis $\{p_\mu(\mathbf{x})\}_{\mu=1}^m$ is employed and the expansion coefficients are found from minimization of weighted squares of residua at a finite number of nodal points (Lancaster and Salkauskas, 1981). The scalar field $u(\mathbf{x})$ can be approximated as

$$u(\mathbf{x}) \approx \sum_{a=1}^{N} \hat{u}^a \phi^a(\mathbf{x}) \tag{14}$$

where N is the total number of nodes, $\hat{u}^a$ is a nodal unknown different from the nodal value $u(\mathbf{x}^a)$, and $\phi^a(\mathbf{x})$ is the shape function associated with the nodal point $\mathbf{x}^a$. Instead of standard MLS-approximation, one can utilize the Central Approximation Node (CAN) concept of MLS-approximation (Sladek et al., 2008). Let $\mathbf{x}^q$ be the CAN for the approximation at a point $\mathbf{x}$. Then, the amount of nodes involved into the approximation at $\mathbf{x}$ is reduced a-priori from N to N^q, where N^q is the number of nodes supporting the approximation at the CAN $\mathbf{x}^q$, i.e. the amount of nodes in the set $M^q = \{\forall\mathbf{x}^a; w^a(\mathbf{x}^q) > 0\}_{a=1}^N$, where

$w^a(\mathbf{x})$ is the weight function associated with the node $\mathbf{x}^a$ at the field point $\mathbf{x}$. In this paper, we employ the Gaussian weights (Sladek et al., 2008). The MLS-CAN approximation is given as

$$u(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\bar{a}} \phi^{(q,a)}(\mathbf{x}) \ , \quad \bar{a} = n(q,a) \tag{15}$$

where $\bar{a}$ is the global number of the a -th node from the N^q nodal points $\mathbf{x}^{\bar{a}} \in M^{\ q}$. The CAN node can be selected as the nearest node to the field point $\mathbf{x}$.

The derivatives of the field variable $u(\mathbf{x})$ can be approximated by differentiating the approximation (15), i.e.

$$u_{,i}(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\bar{a}} \phi_{,i}^{(q,a)}(\mathbf{x}), u_{,ij}(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\bar{a}} \phi_{,ij}^{(q,a)}(\mathbf{x}), u_{,ijk}(\mathbf{x}) \approx \sum_{a=1}^{N^q} \hat{u}^{\bar{a}} \phi_{,ijk}^{(q,a)}(\mathbf{x}) \tag{16}$$

The evaluation of the shape functions and their derivatives at each field point is a numerical procedure which prolongs the CPU time. The relationships required in such numerical evaluations can be found in (Sladek et al., 2012)where Gaussian weights are used in MLS-approximation.

It is worth of consideration the modification of shape functions and their derivatives. Making use the definitions

$$s(\mathbf{x}) := \sum_{a=1}^{N^q} \phi^{(q,a)}(\mathbf{x}), \ s_i(\mathbf{x}) := \sum_{a=1}^{N^q} \phi_{,i}^{(q,a)}(\mathbf{x}),$$

$$s_{ij}(\mathbf{x}) := \sum_{a=1}^{N^q} \phi_{,ij}^{(q,a)}(\mathbf{x}), \ s_{ijk}(\mathbf{x}) := \sum_{a=1}^{N^q} \phi_{,ijk}^{(q,a)}(\mathbf{x}) \tag{17}$$

and adopting the modifications

$$\phi^{(q,a)}(\mathbf{x}) \to \tilde{\phi}^{(q,a)}(\mathbf{x}) := \phi^{(q,a)}(\mathbf{x}) / s(\mathbf{x}),$$

$$\phi_{,i}^{(q,a)}(\mathbf{x}) \to \tilde{\phi}_{,i}^{(q,a)}(\mathbf{x}) := \phi_{,i}^{(q,a)}(\mathbf{x}) - s_i(\mathbf{x}) \tilde{\phi}^{(q,a)}(\mathbf{x}),$$

$$\phi_{,ij}^{(q,a)}(\mathbf{x}) \to \tilde{\phi}_{,ij}^{(q,a)}(\mathbf{x}) := \phi_{,ij}^{(q,a)}(\mathbf{x}) - s_{ij}(\mathbf{x}) \tilde{\phi}^{(q,a)}(\mathbf{x}),$$

$$\phi_{,ijk}^{(q,a)}(\mathbf{x}) \to \tilde{\phi}_{,ijk}^{(q,a)}(\mathbf{x}) := \phi_{,ijk}^{(q,a)}(\mathbf{x}) - s_{ijk}(\mathbf{x}) \tilde{\phi}^{(q,a)}(\mathbf{x}) \tag{18}$$

one can guarantee satisfaction of the following equations

$$\sum_{a=1}^{N^q} \phi^{(q,a)}(\mathbf{x}) = 1 \ , \quad \sum_{a=1}^{N^q} \phi_{,i}^{(q,a)}(\mathbf{x}) = 0 \, ,$$

$$\sum_{a=1}^{N^q} \phi_{,ij}^{(q,a)}(\mathbf{x}) = 0 \ , \quad \sum_{a=1}^{N^q} \phi_{,ijk}^{(q,a)}(\mathbf{x}) = 0 \tag{19}$$

with the wave notation being omitted in Eq. (19) and in what follows.

Point interpolation method (RBF+P)

In this approximation the basis functions are taken as a combination of polynomials and radial basis functions (RBF) (Liu, 2003). Then, one can solve the problem of accuracy and numerical stability of the approximation(Liu, 2003),(Sladek et al., 2008). We shall consider the same polynomial basis as in the MLS-approximation and the RBFs will be taken as multiquadrics

$$R^n(\mathbf{x}) = \left(\left| \mathbf{x} - \mathbf{x}^n \right|^2 + (c^n)^2 \right)^{p/2} \tag{20}$$

with c^n being the shape parameter.

The approximation of the field variable $u(\mathbf{x})$ can be expressed by

$$u(\mathbf{x}) \approx \sum_{a=1}^{N^q} u^{\bar{a}} \varphi^{(q,a)}(\mathbf{x}) \ , \bar{a} = n(q,a) \tag{21}$$

i.e. formally, it is the same as in the MLS-approximation, but now the nodal unknowns are directly the values of the approximated field variable, since the shape functions obey the Kronecker–delta property $\varphi^{(q,a)}(\mathbf{x}^{\bar{b}}) = \delta_{ab}$.

The derivatives of the field variable are approximated as

$$u_{,i}(\mathbf{x}) \approx \sum_{a=1}^{N^q} u^{\bar{a}} \varphi_{,i}^{(q,a)}(\mathbf{x}), \ u_{,ij}(\mathbf{x}) \approx \sum_{a=1}^{N^q} u^{\bar{a}} \varphi_{,ij}^{(q,a)}(\mathbf{x}), \ u_{,ijk}(\mathbf{x}) \approx \sum_{a=1}^{N^q} u^{\bar{a}} \varphi_{,ijk}^{(q,a)}(\mathbf{x}) \tag{22}$$

Again certain numerical procedures are required for evaluation of the shape functions and their derivatives at a field point. For more details, we refer the reader to(Sladek et al., 2012).

5. Rotationally symmetric bending of circular plates

Having regard to the symmetry, the problem simplified when instead of Cartesian coordinates we use polar coordinates (r,φ), where $(x_1,x_2) = (r\cos\varphi, r\sin\varphi)$ and $\partial(.)/\partial\varphi \equiv 0$. Then

$$\begin{aligned} w_{,i}(r) &= r_{,i} \frac{\partial w(r)}{\partial r}, \\ w_{,ij}(r) &= \frac{1}{r}\left(\delta_{ij} - r_{,i} r_{,j}\right) \frac{\partial w(r)}{\partial r} + r_{,i} r_{,j} \frac{\partial^2 w(r)}{\partial r^2}, \\ \nabla^2 w(r) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) w(r) \end{aligned} \tag{23}$$

Since the boundary edge Γ is a circle, the outer unit normal vector on Γ is $n_i = \pm r_{,i}$, where the lower sign is valid on the inner boundary edge in the case of the circular plate with central circular hole. The unit tangent vector on the boundary edge is $t_i = \varepsilon_{3ki} n_k = \pm \varepsilon_{3ki} r_{,k}$. Thus, we may write on the boundary edge

$$M = n_i n_j M_{ij}\Big|_\Gamma = -D r_{,i} r_{,j} \left[(1-\nu) w_{,ij}(r) + \nu \delta_{ij} \nabla^2 w(r) \right]\Big|_\Gamma = -D \left[(1-\nu) \frac{\partial^2 w(r)}{\partial r^2} + \nu \nabla^2 w(r) \right]\Bigg|_\Gamma \tag{24}$$

$$\begin{aligned} T = n_i t_j M_{ij}\Big|_\Gamma &= \mp D \varepsilon_{3lj} r_{,i} r_{,l} \left[(1-\nu) w_{,ij}(r) + \nu \delta_{ij} \nabla^2 w(r) \right]\Big|_\Gamma = \\ &= \mp D (1-\nu) \varepsilon_{3lj} r_{,i} r_{,l} \left[\frac{1}{r}\left(\delta_{ij} - r_{,i} r_{,j}\right) \frac{\partial w(r)}{\partial r} + r_{,i} r_{,j} \frac{\partial^2 w(r)}{\partial r^2} \right]\Bigg|_\Gamma \equiv 0 \end{aligned} \tag{25}$$

$$N = n_i M_{ij,j}\Big|_\Gamma = -Dn_i\left(\nabla^2 w(r)\right)_{,i}\Big|_\Gamma = \mp D\frac{\partial \nabla^2 w(r)}{\partial r}\Big|_\Gamma \tag{26}$$

where we have utilized the fact that

$$\frac{\partial f(r)}{\partial \mathbf{t}}\Big|_\Gamma = \pm\varepsilon_{3ik} r_{,i} f_{,k}(r)\Big|_\Gamma = \pm\varepsilon_{3ik} r_{,i} r_{,k}\frac{\partial f(r)}{\partial r}\Big|_\Gamma \equiv 0\,. \tag{27}$$

Two field variable formulation

The governing equations (10) and (11) can be written now as

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w(r) + m(r) = 0 \text{ or } D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w(r)}{\partial r}\right) + m(r) = 0 \tag{28}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)m(r) = -q(r) \text{ or } \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m(r)}{\partial r}\right) = -q(r) \tag{29}$$

Thus, in view of (24)-(26) and (28), we may write

$$M = \left[m(r) + D(1-\nu)\frac{1}{r}\frac{\partial w(r)}{\partial r}\right]\Big|_\Gamma \tag{30}$$

$$T = 0 \tag{31}$$

$$N = \pm\frac{\partial m(r)}{\partial r}\Big|_\Gamma = V \tag{32}$$

The weak form of the governing equations (28) and (29) is given by

$$D\,r\frac{\partial w(r)}{\partial r}\Big|_{r^c-r_o}^{r^c+r_o} + \int_{r^c-r_o}^{r^c+r_o} r\,m(r)dr = 0 \tag{33}$$

$$r\frac{\partial m(r)}{\partial r}\Big|_{r^c-r_o}^{r^c+r_o} = -\int_{r^c-r_o}^{r^c+r_o} r\,q(r)dr \text{ or } r\frac{\partial m(r)}{\partial r}\Big|_{r^c-r_o}^{r^c+r_o} = -2qr^c r_0, \text{ if } q = \text{const.} \tag{34}$$

From Eqs. (30)-(34), one can see that the weak formulation does not involves higher than first order derivatives of field variables. The numerical results corresponding to this formulation will be denoted by LIE(2xPoiss).

One field variable formulation

In order to see the effect of the proposed decomposition, we present also the standard formulation where all physical quantities are expressed in terms of deflections and their derivatives. The weak form of the governing equation

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w(r) = q(r) \quad \text{or} \quad \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \nabla^2 w(r)}{\partial r}\right) = q(r) \tag{35}$$

is given by

$$Dr\frac{\partial\nabla^2 w(r)}{\partial r}\Bigg|_{r^c-r_o}^{r^c+r_o} = \int_{r^c-r_o}^{r^c+r_o} rq(r)dr\,, \tag{36}$$

Hence, with assuming a constant loading, one obtains

$$D\left(r\frac{\partial^3 w(r)}{\partial r^3} + \frac{\partial^2 w(r)}{\partial r^2} - \frac{1}{r}\frac{\partial w(r)}{\partial r} \right)\Bigg|_{r^c-r_o}^{r^c+r_o} = 2qr^c r_0\ . \tag{37}$$

In the case of symmetric problem for bending of circular plate, the integration can be performed in closed form. One can see that the third order derivatives of the deflection occur in both the weak form of the governing equation and the boundary conditions (on free edge). The numerical results obtained by this formulation will be denoted as LIE(biharmonic).

Recall that exact solutions are available for some simple angular symmetric boundary value problems on circular plate (Sladek et al., 2012).

6. Numerical examples

Several numerical examples will be considered for angularly symmetric bending of circular plate for which exact solutions are available. Then, we can investigate the accuracy and convergence as well as computational efficiency of various presented formulations and techniques. The accuracy of numerical solutions of boundary value problems will be characterized by error norm defined as

$$\text{error norm} = 100\left(\sum_{a=1}^{N}\left[w(r^a) - w^{ex}(r^a) \right]^2 \right)^{1/2} \left(\sum_{a=1}^{N}\left[w^{ex}(r^a) \right]^2 \right)^{-1/2} \tag{37}$$

in the formulation for one field variable, while for two field variables we shall use the definition

$$\text{error norm} = 100\left(\sum_{a=1}^{N}\left\{ \left[w(r^a) - w^{ex}(r^a) \right]^2 + \left[m(r^a) - m^{ex}(r^a) \right]^2 \right\} \right)^{1/2} \left(\sum_{a=1}^{N}\left\{ \left[w^{ex}(r^a) \right]^2 + \left[m^{ex}(r^a) \right]^2 \right\} \right)^{-1/2} \tag{38}$$

where N is the total number of nodal points.

In all numerical computations, we have used a uniform distribution of nodal points and the radius of the sub-domain $r_o = 0.1h$ with h being the distance between two neighbour nodes. The other parameters in the MLS-approximation have been taken as: radius of the interpolation domain $r^a = 3.001h$, shape function parameter $c^a = h$, cubic polynomial basis $m = 4$. In the PIM(RBF+P)-approximation, we have chosen: type of RBF – inverse multiquadrics with $p = -1$, number of multiquadrics around each node $N^q = 16$, number of polynomials $M = 7$, shape parameter $c^a = 2h$. As regards the geometry, we shall consider either the circular plate without any hole $\Omega = \{\forall(r,\varphi); r\in[0,r_a],\ \varphi\in[0,2\pi]\}$ or the circular plate with central hole $\Omega = \{\forall(r,\varphi); r\in[r_b,r_a],\ \varphi\in[0,2\pi]\}$. Three kinds of the boundary value problems will be discussed: (A) $r\in[0,r_a]$, clamped edge (CE) $r = r_a$; (B) $r\in[0,r_a]$, simply supported edge (SSE) $r = r_a$; (C) $r\in[r_b,r_a]$, simply supported edge (SSE) $r = r_b$, free edge $r = r_a$.

Now, we present the results for accuracy and convergence of numerical solutions of three considered boundary value problems with using the MLS-approximations for field variables in two different formulations LIE(biharm) and LIE(2xPoiss). Each formulation is combined with two techniques for creation of shape functions and their derivatives (denoted by S0 and S1). Fig. 1 shows convergence of accuracy of numerical solutions of all considered b.v.p. by LIE(2xPoiss) with increasing the density of nodes (decreasing the h parameter in uniformly distributed nodes). The results are insensitive to the choice of S0 or S1 technique. On the other hand, the LIE(biharm) formulation yields the numerical solutions with unacceptable accuracy and without any indication of convergence (lines with empty symbols in Fig.2). This collapse could be explained by insufficient accuracy of the approximation of $w_{,rrr}(r)$, especially at endpoints $r = r_b$, $r = r_a$. In order to verify this hypothesis, we have modified the LIE(biharm) formulation with replacement of the approximation of $w_{,rrr}(r)$ by exact values $w^{ex}_{,rrr}(r)$. The renewal of convergence of accuracy of numerical solutions (lines with solid symbols in Fig.2) confirms the explanation of the collapse, but there is no proposal how to eliminate this collapse within LIE(biharm).

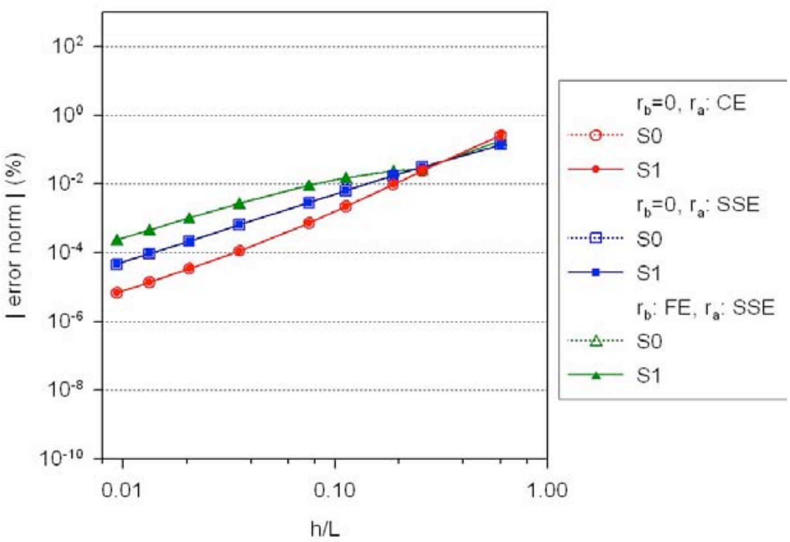

Fig.1 Accuracy and convergence of numerical solutions for three b.v.p. by LIE(2xPoiss) combined with MLS-approximations of field variables

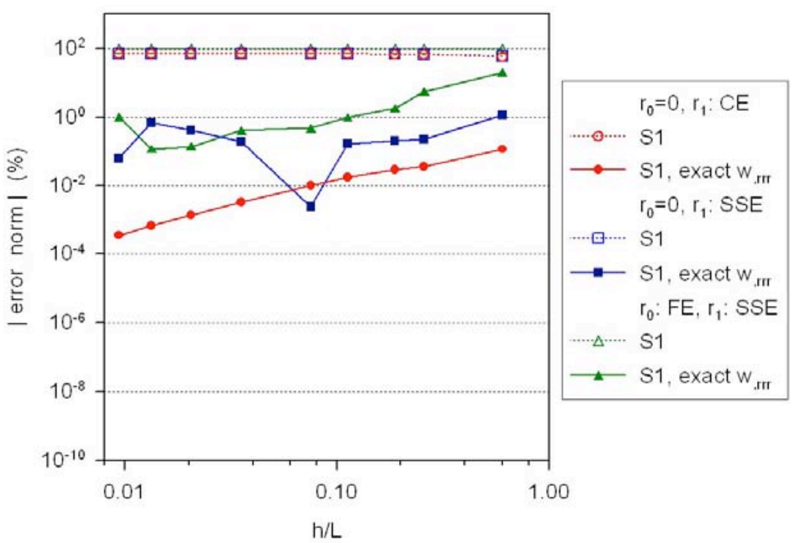

Fig.2 Accuracy and convergence of numerical solutions for three b.v.p. by LIE(biharm) combined with MLS-approximations of field variables

From the above study of the accuracy of numerical solutions by two formulations implemented with MLS-approximations of field variables, the following conclusions can be drawn:

(i) only the formulations for decomposed problem yield meaningful results; the LIE(biharm) formulation fails because of inaccurate approximation of the higher order derivatives of deflections

(ii) the influence of S1-modification for evaluation of shape functions and their derivatives on accuracy is negligible.

Finally, we present the results for numerical solutions of the considered boundary value problems by two discussed formulations but implemented by PIM-approximations of field variables.

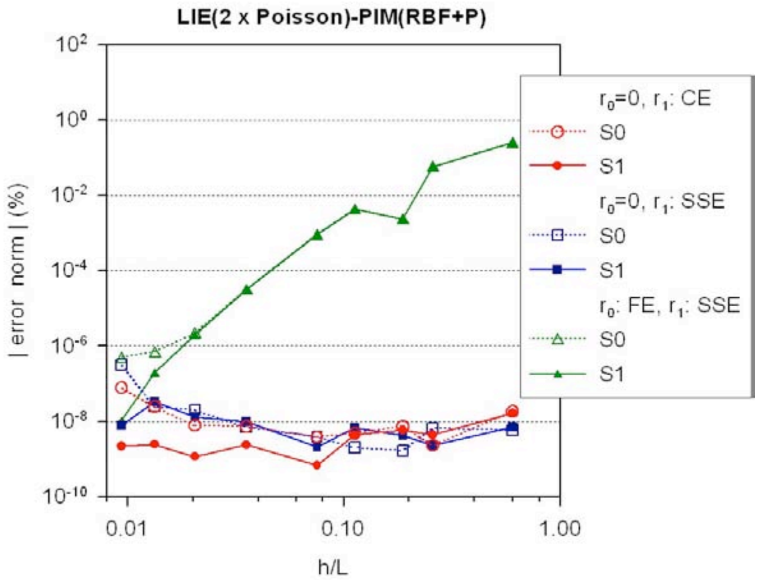

Fig.3 Accuracy and convergence of numerical solutions for three b.v.p. by LIE(2xPoiss) combined with PIM-approximations of field variables

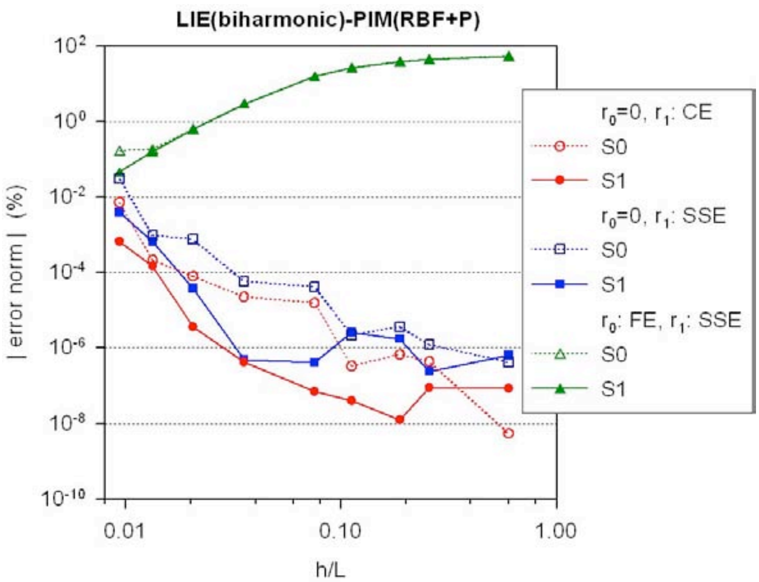

Fig.4 Accuracy and convergence of numerical solutions for three b.v.p. by LIE(biharm) combined with PIM-approximations of field variables

It can be seen from Fig. 3 that in the case of b.v.p. (A) and (B), the accuracy of numerical solutions by LIE(2xPoiss) is very high and more or less invariable with respect to the parameter h in contrast to the case of b.v.p. (C). This can be explained by the detailed study of accuracy of the first order derivatives of the field variables which are almost indifferent to increasing the amount of nodal points only in the case of b.v.p. (A) and (B). The very accurate results are sensitive also to the choice of S0 or S1 technique. Fig. 4 shows that qualitatively different results have been obtained by the LIE(biharm), where the accuracy of the numerical solution of a boundary value problem is affected also by the accuracy of approximations of the third order derivative $w_{,rrr}(r)$. It can be seen that the accuracies of the numerical solutions of b.v.p. (A) and (B) are divergent with respect to decreasing the parameter h, while the convergence of accuracy of the numerical solutions of the b.v.p. (C) is achieved. The detailed study shows that the accuracy of $w_{,rrr}(r)$ is divergent with respect to increasing the amount of nodal points in the b.v.p. (A) and (B) in contrast to improving accuracy of $w_{,rrr}(r)$ in the b.v.p. (C).

Summarizing the study of the accuracy of numerical solutions by two formulations implemented with PIM-approximations of field variables, we conclude:

(i) the LIE(biharm) formulation gives unreliable results (convergence is achieved only in the b.v.p. (C))

(ii) the LIE(2xPoiss) formulation gives stable and highly accurate numerical solutions of the b.v.p. (A) and (B); in the case of b.v.p. (C) the accuracy is good with excellent convergence rate

(iii) the influence of S1-modification for evaluation of shape functions and their derivatives on accuracy is observable as long as the accuracy is very good.

As regards the computational efficiency, the considered computational methods exhibit similar CPU-times (see Fig. 5) when implemented by the MLS- and/or PIM-approximations. Only the LIE(biharm)-MLS spends approximately 3 times shorted computational time than other methods. But this method is disqualified owing to unreliable accuracy.

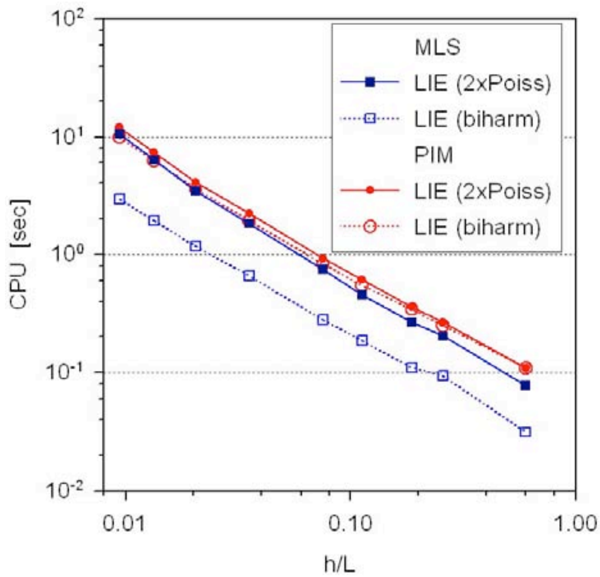

Fig. 5 Comparison of computational times spent by various formulations implemented either by MLS or PIM-approximations

In order to compare the accuracy of two powerful formulations, LIE(2xPoiss)-MLS and LIE(2xPoiss)-PIM, we present following figures with using standard differentiation and S1-technique for evaluation of the shape functions and their derivatives. From Fig. 6, we conclude that much better accuracy is achieved when the formulations are implemented by PIM-approximations of field variables.

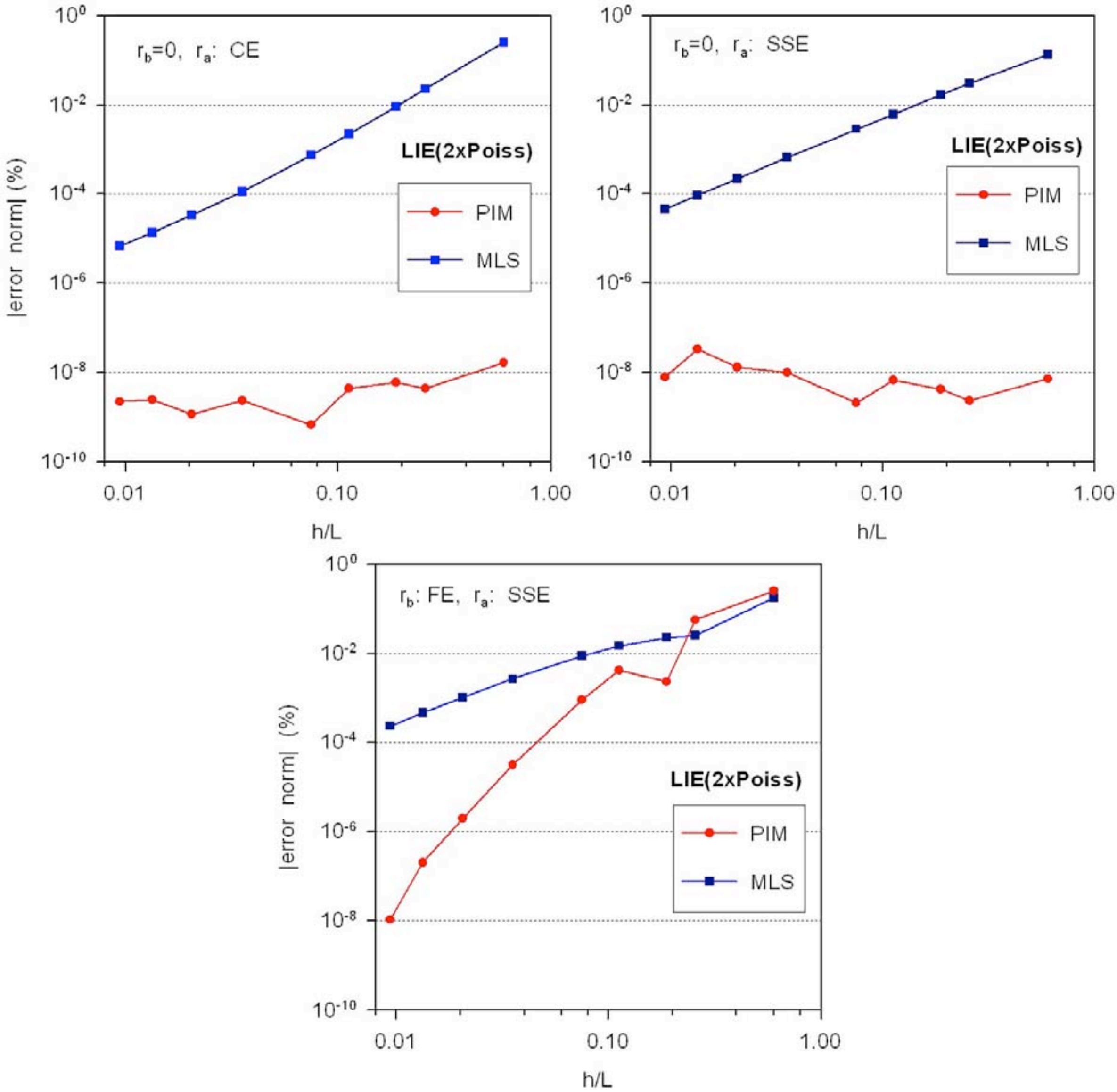

Fig. 6 Comparison of accuracy and convergence of numerical solutions for three kinds of b.v.p. by LIE for decomposed formulations implemented by both PIM- and MLS-approximations

7. Conclusions

The decomposition of the thin plate bending problems governed by the biharmonic operator into two coupled problems governed by Poisson equations is developed and discussed. Two kinds of meshless approximations of field variables are employed in each formulation for numerical solution of boundary value problems. The developed decomposed formulations are not restricted to certain class of boundary value problems. For solution of the decomposed problem, both the strong and local weak formulations have been developed. The accuracy, convergence of accuracy and computational efficiency have been studied for two formulations combined with two meshless approximation of field variables in simple boundary value problems for circular plate. The discussed methods give reasonable numerical results when applied to decomposed problem, while the methods applied to original biharmonic problem fail.

Acknowledgements

This work was partially supported by the Slovak Research and Development Agency under the contract No. APVV-0032-10. This article has been produced also with the financial assistance of the European Regional Development Fund (ERDF) under the Operational Programme Research and Development/Measure 4.1 Support of networks of excellence in research and development as the pillars of regional development and support to international cooperation in the Bratislava region/Project No. 26240120020 Building the centre of excellence for research and development of structural composite materials – 2nd stage**.**

References

Lancaster P., Salkauskas K. (1981) Surfaces generated by moving least square method. *Mathematics of Computations*; **3**7: 141-158.

Liu GR. (2003) Mesh Free Methods, Moving Beyond the Finite Element Method. CRC Press: Boca Raton.

Sladek V, Sladek J, Zhang Ch. (2008) Computation of stresses in non-homogeneous elastic solids by local integral equation method: a comparative study. *Computational Mechanics*, **41**: 827-845.

Sladek V, Sladek J, Tanaka M. (2005) Local integral equations and two meshelss interpolations with application to potential problems in non-homogeneous media. *CMES - Computer Modeling in Engineering & Sciences*; **7**: 69-83.

Sladek, V., Sladek, J., Sator, L. (2012) Physical decomposition of thin plate bending problems and their solution by mesh-free methods, submitted for publication

Timoshenko S.P., Woinowsky-Krieger S. (1959) *Theory of Plates and Shells*. McGraw-Hill: New York.