17th International Conference
ENGINEERING MECHANICS 2011
Svratka, Czech Republic, 9 – 12 May 2011

APPLICATION OF MESHLESS MLPG METHOD FOR TRANSIENT ANALYSIS OF AXISYMMETRIC CIRCULAR PLATE BENDING PROBLEM

P. Staňák[*], J. Sládek[*], V. Sládek[*]

Abstract: *Axisymmetric circular plates subjected to stationary and transient dynamic loads are analyzed in the presented paper. Effect of viscous damping is also considered. Dynamic loading with impact and stepped time history is taken into account. The governing equation for the bending of plate represented by partial differential equation (PDE) of the fourth order is decomposed into two coupled PDEs of the second order. Clamped plate edge as a boundary condition is assumed. Axisymmetric assumptions reduce the problem to one dimensional. Each node is a center of 1-D interval subdomain. The weak-form on these small subdomains is applied to derive local integral equations with a unit step function as the test function. Moving least-squares (MLS) approximation technique is applied to obtain system of ordinary differential equations (ODE). Houbolt finite difference scheme is finally applied to solve this system of ODE for certain nodal unknowns.*

Keywords: ***Local integral equations, meshless approximation, Kirchhoff plate theory, Houbolt finite-difference scheme.***

1. Introduction

Analysis of circular plates is basic problem in structural mechanics, as long as many structures, like containers or reservoirs, have circular basement. Another reason why to deal with circular plates is the availability of exact solution that is convenient for assessment of new numerical techniques.

The MLPG (Meshless Local Petrov-Galerkin) method (Atluri, 2004) is one of the most rapidly developing meshless method. The MLPG method was applied to static and dynamic loading of thin circular and square plates (Sladek et al., 2003), also to dynamic loading of thick Reissner-Mindlin plates (Sladek et al., 2007). Recently, MLPG method was applied also to laminated composite plates (Sladek et al., 2010a) and piezoelectric plates (Sladek et al., 2010b).

Presented approach reduces the problem of bending of circular plate to 1-D with the assumption of axisymmetric conditions. The governing equation for for plate bending problem is decomposed to reduce the order of differentiation. Application of MLS approximation to derive local integral equations leads to the system of ordinary differential equations that is solved by the Houbolt method.

2. Local integral equations for an axisymmetric circular plate

Let us consider a homogeneous axisymmetric clamped circular plate with the radius *r* and thickness *h* occupying the domain Ω. The plate is subjected to the transverse dynamic load $q(\mathbf{x},t)$. According to the classical (Kirchhoff) plate theory the differential equation for the plate deflection $w(\mathbf{x},t)$ can be written in the form

$$D\nabla^2\nabla^2 w(\mathbf{x},t) + \rho h \ddot{w}(\mathbf{x},t) + g\dot{w}(\mathbf{x},t) = q(\mathbf{x},t) \qquad (1)$$

[*] Ing. Peter Staňák, prof. Ing. Ján Sládek, DrSc. and prof. RNDr. Vladimír Sládek, DrSc.: Institute of Construction and Architecture, Slovak Academy of Sciences, Dúbravská cesta 9; 845 03, Bratislava; Slovakia, e-mails: peter.stanak@savba.sk, jan.sladek@savba.sk, vladimir.sladek@savba.sk

where ρ is the mass density, g is the viscous damping coefficient and $D = Eh^3/12(1-\nu^2)$ is the plate stiffness, with E as Young's modulus and ν denoting Poisson's ratio. The dots over the quantity represent differentiation with respect to time *t*.

It is possible to decrease the order of differentiation in eq. (1) by its decomposition into two PDEs of the second order (Sladek et al., 2003) as

$$-D\nabla^2 w(\mathbf{x},t) = m(\mathbf{x},t) \tag{2}$$

$$\nabla^2 m(\mathbf{x},t) - \rho h\ddot{w}(\mathbf{x},t) - g\dot{w}(\mathbf{x},t) = -q(\mathbf{x},t) \tag{3}$$

where the quantity $m(\mathbf{x},t)$ is proportional to the spur of the bending moment tensor as $m = M_{ii}/(1+\nu)$ (Balas et al., 1989). On a clamped edge of the plate, however, $m(\mathbf{x},t)$ is equal to the bending moment $M(\mathbf{x},t)$. For a simply supported edge this is only valid if the edge is straight (Sladek et al., 2003). For a circular plate with curved edges the quantity $m(\mathbf{x},t)$ has no physical interpretation and the definition of boundary conditions is impossible. This is the main reason why only clamped circular plates will be analyzed. Note that Eqs. (2), (3) are coupled and that is why they must be solved simultaneously. If we restrict ourselves to symmetric material properties, loading and boundary conditions throughout the plate, the problem can be simplified to axisymmetric case in polar coordinates. Under axisymmetric conditions all the variables in Eqs. (2) and (3) will be function of the radial coordinate *r* only.

The MLPG method is based on the local weak form of the governing equations. The local weak form for Eqs. (2), (3) is then written over a small local subdomain Ω_s (Atluri, 2004) as

$$\int_{\Omega_s} \left[D\nabla^2 w(r,t) + m(r,t) \right] h^*(r)\, d\Omega = 0 \tag{4}$$

$$\int_{\Omega_s} \left[\nabla^2 m(r,t) - \rho h\ddot{w}(r,t) - g\dot{w}(r,t) + q(r,t) \right] h^*(r)\, d\Omega = 0 \tag{5}$$

where $h^*(r)$ is the weight or test function. In the present analysis unit step function is used, as defined by Sladek et al. (2010a). Let *w* and *m,* hereafter called the trial functions, be an approximate solution to the problem. Integrating Eqs. (4), (5) by parts with assumptions of Laplace operator ∇^2 in polar coordinates, considering the integration element $d\Omega = rdr$ for the axisymmetric circular plate and making use of the unit step weight function defined over Ω_s is leading to local integral equations for the presented problem. This process can be also observed in papers by Sladek et al. (2003, 2010a).

The MLS (Moving least-squares) approximation (Lancaster & Salkaustas, 1981; Atluri, 2004) can be used for the approximation of unknown quantities in terms of the nodal values as

$$\mathbf{w}^h(r,\tau) = \sum_{a=1}^{n} \phi^a(r)\hat{\mathbf{w}}^a(\tau) \quad , \quad \mathbf{w}_{,r}{}^h(r,\tau) = \sum_{a=1}^{n} \phi_{,r}{}^a(r)\hat{\mathbf{w}}^a(\tau) \tag{6}$$

where $\phi^a(r)$, $\phi_{,r}{}^a(r)$ are called MLS shape functions for unknown quantity and its derivative. Applying Eq. (6) for approximation of trial functions $w(r,t)$, $m(r,t)$ and their derivatives with subsequent introduction into local integral equations gives the discretized local integral equations

$$\sum_{i=1}^{n} \left[D\left(r\phi_{,r}^i(r) \right)\Big|_a^b \right] \hat{w}^i(t) + \sum_{i=1}^{n} \left[\int_{\Omega_s} r\phi^i(r)\, dr \right] \hat{m}^i(t) = 0 \tag{7}$$

$$\sum_{i=1}^{n} \left[\left(r\phi_{,r}^i(r) \right)\Big|_a^b \right] \hat{m}^i(t) - \sum_{i=1}^{n} \left[\rho h \int_{\Omega_s} r\phi^i(r)\, dr \right] \hat{\ddot{w}}^i(t) - \sum_{i=1}^{n} \left[g \int_{\Omega_s} r\phi^i(r)\, dr \right] \hat{\dot{w}}^i(t) = -\int_{\Omega_s} qrdr \tag{8}$$

To impose boundary conditions, method of Lagrange multipliers or penalty method can be used in most of the numerical solution methods. MLPG, however, allows us to use also collocation approach to impose boundary conditions directly, using interpolation approximations (6) as $\sum_{i=1}^{n} \phi^i(r^j)\hat{w}^i(\tau) = \tilde{w}^i(r^j,\tau)$ where $\tilde{w}^i(r,\tau)$ is the prescribed value of the deflection on the boundary. Similar approach can be adopted for the quantity $m(r,t)$. Collecting the discretized local boundary-domain integral equations together with the discretized boundary conditions for the generalized displacements, one obtains a complete system of ordinary differential equations (ODE) which can be rearranged in such a way that all known quantities are on the r.h.s. Thus, in the matrix form the system becomes

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{Y} \tag{9}$$

This system of ODE (9) can be conveniently solved by the Houbolt finite-difference scheme (Houbolt, 1950; Sladek et al., 2010a). This method is strongly dependent on the size of the time step. The value of the time step must be appropriately selected with respect to material parameters and time dependence of boundary conditions.

3. Numerical examples

Let us consider a circular plate with the radius $r_0 = 0.5$ m and the thickness $h = 0.002$ m. Material properties are as follows: Young's modulus $E = 2\times10^{11}\,\mathrm{Nm}^{-2}$, Poisson's ratio $\nu = 0.3$ and mass density $\rho = 7850\,\mathrm{kg.m}^{-3}$. Two load cases are considered, a uniform static loading with q = 10 Pa and dynamic loading with the Heaviside time variation with amplitude q = 10 Pa. 21 equally-spaced nodes are used for the discretization of plate geometry. For the clamped axisymmetric circular plate under uniform static loading the exact solution is given as

$$w(r) = \frac{qr_0^4}{64D}\left[1-\left(\frac{r}{r_0}\right)^2\right]^2 \tag{10}$$

Assuming $r = r_0$ in Eq. (10) and inserting all quantities, the exact deflection of the plate center is obtained as $w(r = r_0) = 0.667e^{-4}$ m. The variation of the deflection with the radial coordinate is presented in Fig. 1. The results are in excellent agreement.

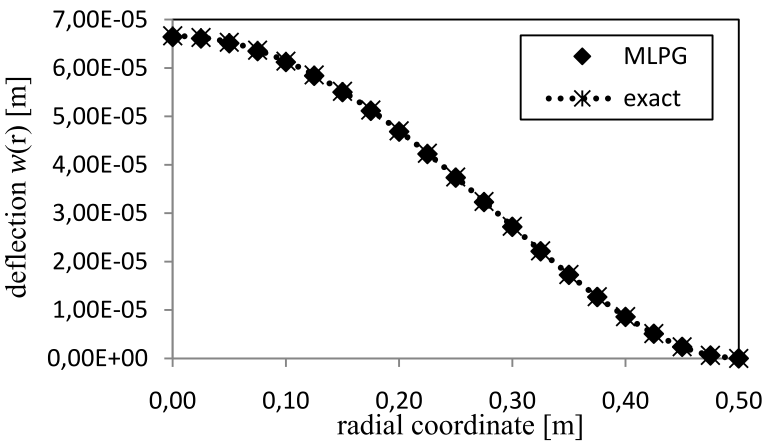

Fig. 1: Variation of the deflection with radial coordinate for clamped axisymmetric circular plate.

Next dynamic loading is considered with the Heaviside time variation. Again 21 equally distributed nodes are used with radius of support domain with size of 10 times nodal distance of two neighboring nodes. Numerical calculations were carried out for a time step $\Delta t = 0.8e^{-3}$ s with 200 time increments. Viscous damping is also considered. It is defined through the damping parameter $\xi = g/g_c = 0.1$, where the critical damping is $g_c = 2\omega_1\rho h$, with $\omega_1 = 125$ rad/s being the first natural frequency. Variation of the center point deflection with time is presented in Fig. 2. Results from MLPG analysis are compared to transient FEM analysis with the same geometry and mesh as in the previous static case. A quarter of the plate is modeled due to symmetry in FEM/ANSYS code with fine mesh. Again, excellent match of the result is observed.

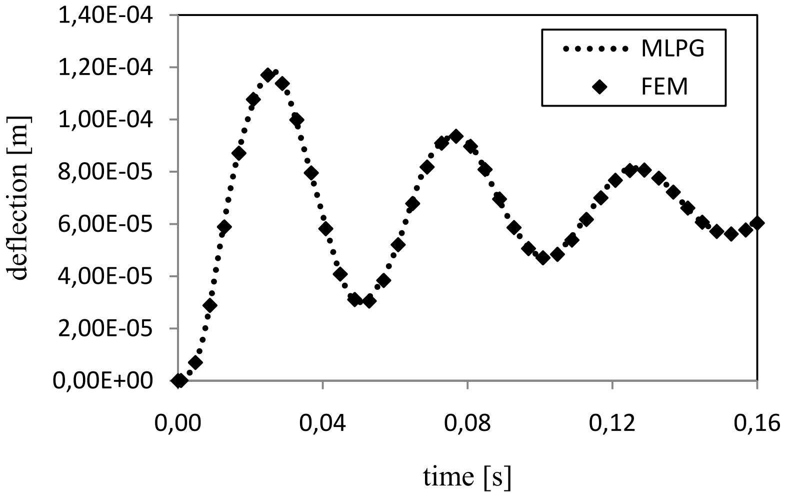

Fig. 2: Variation of deflection at center of the considered plate with respect to time.

4. Conclusion

The MLPG method is presented for solving bending problems of thin axisymmetric circular plates. Both static and dynamic loads are considered. The analysed domain is divided into small overlapping subdomains. A unit step function is used as the test function in the local weak form. The MLS approximation scheme is adopted for approximation of unknown physical quantities. The proposed method is a truly meshless method as it requires no background mesh in neither interpolation, nor integration. It can be considered as an alternative to many existing computational methods.

Acknowledgement

The support of the grant VEGA-2/0039/09 by Slovak Grant Agency is gratefully acknowledged.

References

Atluri, S. N. (2004) The meshless method (MLPG) for domain & BIE discretizations. Tech Science Press, Forsyth, USA.

Balas, J., Sladek, J. & Sladek, V. (1989) Stress analysis by boundary element methods. Elsevier, Amsterdam.

Houbolt, J. C. (1950) A recurrence matrix solution for the dynamic response of elastic aircraft. Journal of Aeronautical Sciences, 17, pp. 371-376.

Lancaster, P., Salkauskas, T. (1981) Surfaces generated by moving least square methods. Math. Comput., 37, pp. 141-158.

Sladek, J., Sladek, V., Krivacek, J., Wen, P. & Zhang, Ch. (2007) Meshless Local Petrov-Galerkin (MLPG) method for Reissner-Mindlin plates under dynamic load. Computer Meth Appl Mech Engn, 196, pp. 2681-2691.

Sladek, J., Sladek, V. & Mang, H. A. (2003) Meshless LBIE formulations for simply supported and clamped plates under dynamic load. Computers and Structures, 81, pp. 1643-1651.

Sladek, J., Sladek, V., Stanak, P., Zhang, Ch. (2010a) Meshless Local Petrov-Galerkin (MLPG) Method for Laminate Plates under Dynamic Loading. CMC- Computers, Materials & Continua, 15, pp. 1-26.

Sladek, J., Sladek, V., Stanak, P., Pan, E. (2010b) The MLPG for bending of electroelastic plates. CMES-Computer Modeling in Engineering & Sciences, 64, pp. 267-298.