17th International Conference
ENGINEERING MECHANICS 2011
Svratka, Czech Republic, 9 – 12 May 2011

ON APPLICATIONS OF GENERALIZED FUNCTIONS TO CALCULATION OF THIN CYLINDRICAL SHELLS

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Abstract: *The mathematical model of a thin cylindrical shell according to Timoshenko and Love bending theory contains derivatives of generalized internal forces and deformation components. However these derivatives are not defined at such points between ends of the shell where a concentrated loading or an internal support or coupling is located. In order that the mathematical model of a thin cylindrical shell subjected to axisymmetric loading may hold true at the points of discontinuity mentioned, which are common in calculating experience, we have used the distributional derivative for the unknown quantities, and developed a generalized mathematical model in the form of a system of ordinary differential equations (SODE). We have found the general solution to the SODE by using the Laplace transform method and symbolic programming approach. The solution found is a generalization of Krylov functions method.*

Keywords: Thin cylindrical shell, discontinuities, Dirac singular distribution, Heaviside step function.

1. Introduction

Solving analytically a thin cylindrical shell subjected to an axisymmetric bending with discontinuous loading, support or geometry, we at first divide it into segments without discontinuities. Then, we find continuous solutions with integration constants for each shell segment separately. Finally, we determine integration constants using boundary conditions and continuity conditions among shell segments.

Applying distributional derivative (Schwartz, 1966) for the transverse shear force per unit length, the axial bending moment per unit length, and for the angle of rotation of a middle-surface normal in a meridian plane, we can derive a generalized mathematical model of the thin-walled cylindrical shell with discontinuities in loading, support and geometry that may be solved like only one differential problem without dividing shells into cylindrical segments, and without using continuity conditions.

2. The classical mathematical model of an axisymmetric bending of the thin cylindrical shell

According to the Love-Timoshenko shell bending theory (Love, 1944; Timoshenko, 1959), a system of differential equations describing axisymmetric bending of thin cylindrical shells may be composed of four ordinary differential equations of the first order (Höschl, 1971; Markuš, 1982; Němec et al., 1989) as follows

$$\frac{d}{dx}\mathrm{T}(x) = p_n(x) - \frac{N_t(x)}{r}, \tag{1}$$

$$\frac{d}{dx}M_a(x) = \mathrm{T}(x), \tag{2}$$

$$\frac{d}{dx}\phi(x) = \frac{M_a(x)}{\mathrm{D}}, \tag{3}$$

$$\frac{d}{dx}\mathrm{w}(x) = \phi(x), \tag{4}$$

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for

$$\left(\frac{d}{dx}\mathrm{w}(x)\right)^2 \ll 1 \quad , \quad \frac{\mathrm{w}(x)}{r^2} \approx 0, \tag{5}$$

where

$\mathrm{T}(x)$	transverse shear force per unit length,
$M_a(x)$	axial bending moment per unit length,
$\phi(x)$	angle of rotation of a middle-surface normal in a meridian plane,
$\mathrm{w}(x)$	radial displacement component,
$p_n(x)$	surface loading in normal direction to the middle surface,
$N_t(x)$	membrane normal force per unit length in the circumferential direction,
r	radius of the middle surface of the shell,
D	shell wall bending stiffness,
x	axial coordinate.

Equations (1), (2) are equilibrium conditions of a shell element cut out in the undeformed shape. The equation (3) expresses relationship between the axial bending moment per unit length and a middle-surface curvature change supposing (5). Membrane normal force per unit length acting in the circumferential direction may be expressed as follows

$$N_t(x) = \frac{E\,\mathrm{w}(x)\,h}{r} + \mu\,N_a\,, \tag{6}$$

where

E	Young’s modulus,
μ	Poisson’s ratio,
h	thickness of the shell wall,
N_a	axial membrane normal force per unit length.

3. The generalized mathematical model of axisymmetric bending of the thin cylindrical shell

Concentrated radial circumferential-line force loadings per unit length and concentrated supports along circumferential lines situated between ends of the shell may cause jump discontinuities of the transverse shear force per unit length. Concentrated circumferential-line moment loadings placed between ends of the shell may cause jump discontinuities of the bending moments per unit length. Circumferential hinges connecting cylindrical shell segments may cause jump discontinuities of the angle of rotation of the middle-surface normal in a meridian plane.

Classical derivatives (1) to (3) do not hold true at points of the jump discontinuities mentioned because quantities $\mathrm{T}(x)$, $M_a(x)$ and $\phi(x)$ were supposed to be continuous. In order to remove this inconsistency, we have used the distributional derivative (Kanwal, 2004; Štěpánek, 2001) for discontinuous unknown quantities $\mathrm{T}(x)$, $M_a(x)$, $\phi(x)$, and regarding (6), we have derived the following generalized mathematical model

$$\frac{d}{dx}\mathrm{T}(x) = p_n(x) - \frac{\mu\,N_a}{r} - \frac{E\,\mathrm{w}(x)\,h}{r^2} + \left(\sum_{i=1}^{n_1} F_i\,\mathrm{Dirac}(x-a_i)\right) - \left(\sum_{i=1}^{n_2} R_i\,\mathrm{Dirac}(x-b_i)\right), \tag{7}$$

$$\frac{d}{dx}M_a(x) = \mathrm{T}(x) + \left(\sum_{i=1}^{n_3} C_i\,\mathrm{Dirac}(x-g_i)\right), \tag{8}$$

$$\frac{d}{dx}\phi(x) = \frac{M_a(x)}{D} + \left(\sum_{i=1}^{n_4} \Phi_i \, \mathrm{Dirac}(x - k_i)\right), \tag{9}$$

$$\frac{d}{dx}\mathrm{w}(x) = \phi(x), \tag{10}$$

where

F_i	i-th concentrated radial circumferential-line force loading per unit length,
R_i	i-th concentrated radial circumferential-line reaction force per unit length,
C_i	i-th concentrated circumferential-line moment loading per unit length,
Φ_i	magnitude of a jump discontinuity of $\phi(x)$ at i-th circumferential hinge connection of shell segments,
n_1	number of concentrated radial circumferential-line force loadings,
n_2	number of concentrated radial circumferential-line reaction forces,
n_3	number of concentrated circumferential-line moment loadings,
n_4	number of circumferential hinge connections of cylindrical shell segments,
Dirac(x-x_0)	Dirac singular distribution moved to $x = x_0$, $x_0 > 0$.

4. The general solution to the generalized system of differential equations (7) to (10)

First, we introduce an auxiliary constant as follows

$$\Omega^4 = \frac{E\,h}{4\,r^2\,D}, \tag{11}$$

where the shell wall bending stiffness is

$$D = \frac{E\,h^3}{12\,(1-\mu^2)}. \tag{12}$$

Applying the Laplace transformation to equation (7) to (10) with respect to x, we obtain an algebraic system from which we can find the Laplace transforms of all four unknown quantities. Regarding the short extent of this paper, we have presented here only the Laplace transform of the radial displacement component as follows

$$\begin{aligned}
\mathrm{laplace}(\mathrm{w}(x), x, p) = {} & \frac{p^3\,\mathrm{w}(0)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)} + \frac{p^2\,\phi(0)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)} \\
& + \frac{p\,M_a(0)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D} + \frac{\mathrm{T}(0)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D} \\
& + \frac{\sum_{i=1}^{n_1} \mathrm{e}^{(-p\,a_i)}\,F_i}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D} - \frac{\sum_{i=1}^{n_2} \mathrm{e}^{(-p\,b_i)}\,R_i}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D} \\
& + \frac{p\left(\sum_{i=1}^{n_3} \mathrm{e}^{(-p\,g_i)}\,C_i\right)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D} + \frac{p^2\left(\sum_{i=1}^{n_4} \mathrm{e}^{(-p\,k_i)}\,\Phi_i\right)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)} \\
& - \frac{\mu\,N_a}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D\,r\,p} + \frac{\mathrm{laplace}(p_n(x), x, p)}{(2\,\Omega^2 - 2\,\Omega\,p + p^2)\,(2\,\Omega^2 + 2\,\Omega\,p + p^2)\,D}
\end{aligned} \tag{13}$$

where p is a complex variable. Converting the right side of (13) into partial fractions and applying the inverse Laplace transformation, we can express the radial displacement component as follows

$$
\begin{aligned}
w(x) &= \cosh(\Omega x)\cos(\Omega x)\,w(0) + \frac{(\sinh(\Omega x)\cos(\Omega x) + \cosh(\Omega x)\sin(\Omega x))\,\phi(0)}{2\,\Omega} \\
&+ \frac{1}{2}\frac{\sin(\Omega x)\sinh(\Omega x)\,M_a(0)}{\Omega^2\,D} + \frac{1}{8}\frac{(2\cosh(\Omega x)\sin(\Omega x) - 2\sinh(\Omega x)\cos(\Omega x))\,T(0)}{\Omega^3\,D} \\
&+ \frac{\sum_{i=1}^{n_1} \mathrm{Heaviside}(x-a_i)\,(\cosh(\Omega(x-a_i))\sin(\Omega(x-a_i)) - \sinh(\Omega(x-a_i))\cos(\Omega(x-a_i)))\,F_i}{4\,\Omega^3\,D} \\
&+ \frac{\sum_{i=1}^{n_2} (-\cosh(\Omega(x-b_i))\sin(\Omega(x-b_i)) + \sinh(\Omega(x-b_i))\cos(\Omega(x-b_i)))\,\mathrm{Heaviside}(x-b_i)\,R_i}{4\,\Omega^3\,D} \\
&+ \frac{\sum_{i=1}^{n_3} \sinh(\Omega(x-g_i))\sin(\Omega(x-g_i))\,\mathrm{Heaviside}(x-g_i)\,C_i}{2\,\Omega^2\,D} \\
&+ \frac{\sum_{i=1}^{n_4} \mathrm{Heaviside}(x-k_i)\,\Phi_i\,(\sinh(\Omega(x-k_i))\cos(\Omega(x-k_i)) + \cosh(\Omega(x-k_i))\sin(\Omega(x-k_i)))}{2\,\Omega} \\
&+ \frac{1}{4}\frac{\mu\,(-1 + \cosh(\Omega x)\cos(\Omega x))\,N_a}{\Omega^4\,D\,r} \\
&+ \int_0^x \frac{1}{4}\frac{p_n(\xi)\,(\cosh(\Omega(x-\xi))\sin(\Omega(x-\xi)) - \sinh(\Omega(x-\xi))\cos(\Omega(x-\xi)))}{\Omega^3\,D}\,d\xi
\end{aligned}
\tag{14}
$$

5. Conclusions

The contribution of this paper is that the generalized mathematical model of a thin cylindrical shell (7) to (10) holds true also for discontinuous graphs of the transverse shear force per unit length, the axial bending moment per unit length, and the angle of rotation of the middle-surface normal in a meridian plane caused by concentrated radial circumferential-line force loadings, concentrated supports along circumferential lines, concentrated circumferential-line moment loadings situated between ends of the shell, and circumferential hinges connecting cylindrical shell segments. The jump discontinuities of the unknown quantities have been expressed using Dirac singular distribution at the right side of Eq. (7) to (9). In order to determine magnitudes of the unknown jump discontinuities owing to the supports or hinges between ends of the shell, we have to use deformation conditions at points of these discontinuities. The general solution to the Eq. (7) to (10) has been computed using the Laplace transform method and symbolic programming approach, and has been partly presented in (14). The integration constants have got the form of initial parameters, and may be determined using boundary conditions. The jump discontinuities of the axisymmetric radial surface loading pn(x) may be expressed using Heaviside step function.

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