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THE ANALYSIS OF THE CRACK INITIATED FROM THE ORTHOTROPIC BI-MATERIAL SHARP NOTCH

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Abstract: *Under the consideration of the bi-material notch composed of two orthotropic materials the potential direction of the crack initiated from the notch tip is determined from the maximum mean value of the tangential stresses and local minimum of the mean value of the generalized strain energy density factor in both materials. Following the assumption of the same mechanism of the rupture in the case of the crack and the notch, an expression for the critical values of the generalized stress intensity factor can be obtained. The radial and tangential stresses and strain energy density are expressed using the Lekhnitskii-Eshelby-Stroh (LES) formalism for the plane elasticity. The stress singular exponents and corresponding eigenvectors are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions. In generally, there is more than one solution of this eigenvalue problem and consequently the generalized stress intensity factors.*

Keywords: ***Orthotropic, bi-material, sharp notch, LES formalism, Ψ-integral, MTS theory.***

1. Stress distribution

In the contribution the orthotropic bi-material notch is analyzed from the perspective of generalized linear elastic fracture mechanics, i.e. the validity of small-scale yielding conditions is assumed. It is further assumed ideal adhesion at the bi-material interface and the notch radius r→0 (the sharp bi-material notch tip), see Fig. 1. The necessary step for the crack initiation assessment is detailed knowledge of the stress distribution. Within plane elasticity of anisotropic media the Lekhnitskii-Eshelby-Stroh (LES) formalism based on Lekhnitskii (1963) can be used. The relations for displacements and stresses can be found in Klusák et al. (2010) and Profant et al. (2010). In the case of the studied notch, the potentials $f_j(z_j)$ appearing in the LES formalism have the following form $\mathbf{f} = H\left\langle z_*^{\delta}\right\rangle \mathbf{v}$, where H is the generalized stress intensity factor, $v_i = v_i' + \mathrm{i}v_i''$ is an eigenvector corresponding to the eigenvalue $\delta = \delta' + \mathrm{i}\delta''$ representing the exponent of the stress singularity at the notch tip p via the relation $p = \delta - 1$. The expression $\left\langle z_*^{\delta}\right\rangle$ represents the diagonal matrix with

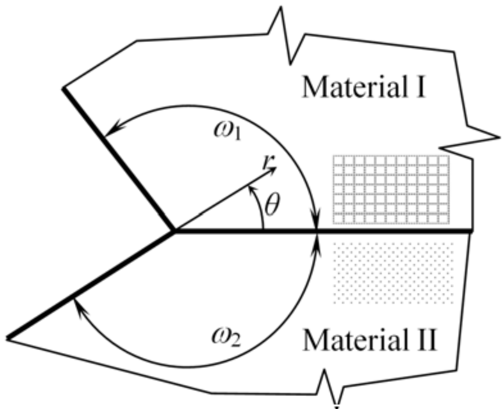

Fig. 1: Bi-material orthotropic notch with corresponding polar coordinate system.

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diagonal elements $z_i = x + \mu_i y$, where $i = 1, 2$ and μ_i is an eigenvalue of the material, Lekhnitskii (1963), Klusák et al. (2010) and Profant et al. (2010). Eigenvector v_i and eigenvalue δ are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions.

2. GSIFs, crack initiation direction and stability criterion

***Generalized stress intensity factors determination*:** The GSIFs can be determined using the so-called Ψ - integral, Klusák et al. (2010) and Profant et al. (2010). This method is an implication of Betti's reciprocity theorem which in the absence of body forces states that the following integral is path-independent

$$\Psi(\mathbf{u},\hat{\mathbf{u}}) = \int_\Gamma (\sigma_{ij}(\mathbf{u}) n_i \hat{u}_j - \sigma_{ij}(\hat{\mathbf{u}}) n_i u_j) \mathrm{d}s \quad (i, j = 1, 2). \tag{1}$$

The contour Γ surrounds the notch tip and the displacements u_j are considered as the regular and $\hat{u}_j$ as the auxiliary solutions of the eigenvalue problem of the notch.

***Mean value of the tangential stress*:** For mixed mode fields a crack may grow along the interface or at a certain angle θ_0 with the interface into the material I or II. The criterion of the MTS theory, Erdogan & Sih (1963), states that the crack is initiated in the direction θ_0 where the circumferential stress $\sigma_{\theta\theta}$ at some distance from the crack tip has its maximum and reaches a critical tensile value. In order to suppress the influence of the distance *r*, the mean value of the tangential stress is evaluated over a certain distance *d* and the potential direction of the crack initiation is determined from the maximum of this mean value of tangential stress in both materials. Hence the following realations have to be satisfied

$$\left(\frac{\partial \overline{\sigma}_{m\theta\theta}}{\partial \theta}\right)_{\theta=\theta_0} = 0, \quad \left(\frac{\partial^2 \overline{\sigma}_{m\theta\theta}}{\partial \theta^2}\right)_{\theta=\theta_0} < 0, \quad \overline{\sigma}_{\theta\theta}(\theta) = H_1 F_{\theta\theta 1m}(\theta, d) + H_2 F_{\theta\theta 2m}(\theta, d), \tag{2}$$

where H_i are generalized stress intensity faktors and functions $F_{\theta\theta im}(\theta, d)$ can be found in Profant et al. (2010).

***Strain energy density factor*:** Similarly, the crack initiation direction can be derived via the generalization of the mean value of the strain energy density factor $\overline{\Sigma}_m(\theta, d)$ defined in Profant et al. (2010). To find the minimum of $\overline{\Sigma}_m(\theta, d)$ and consequently the crack initiation direction in materials I or II, following conditions have to be determined

$$\left(\frac{\partial \overline{\Sigma}_m}{\partial \theta}\right)_{\theta=\theta_0} = 0, \quad \left(\frac{\partial^2 \overline{\Sigma}_m}{\partial \theta^2}\right)_{\theta=\theta_0} > 0, \tag{3}$$

$$\overline{\Sigma}_m(\theta, d) = \frac{1}{4\pi d} \int_0^d \left(r^{2\delta_1'-1} H_1^2 U_{1m} + r^{2\delta_2'-1} H_2^2 U_{2m} + 2 r^{\delta_1'+\delta_2'-1} H_1 H_2 U_{12m} \right) dr. \tag{4}$$

The function $U_{ijm}(\theta)$ as well as the functions $F_{\theta\theta im}(\theta, d)$ hold an complicated form and can be found in Profant et al. (2010).

***Stability criterion*.** The stability criterion of the orthotropic bimaterial notches defines the loading conditions above that a crack is initiated in the tip of the singular stress concentrator. Consider the fact that for the GSIFs and their critical values the folowing condition holds, $\Gamma_{21} = H_2 / H_1 = H_{2C} / H_{1C}$, and that the boundary conditions does not depend on the absolute value of the applied stress σ_{appl}.

The assumption of the same mechanism of a rupture of the crack and the notch leads to the expression of the critical value of H_1, Klusák et al. (2010) and Profant et al. (2010),

$$H_{1C} = \frac{2K_{IC}}{\sqrt{2\pi d}\left(F_{\theta\theta 1m}(\theta_0) + \Gamma_{21} F_{\theta\theta 2m}(\theta_0)\right)}, \tag{5}$$

where K_{IC} is the fracture toughness of the material *m*.

***Stability of the notch - Matched asymptotic procedure*:** Matched asymptotic procedure is used to derive the change of potential energy for the debonding crack and the crack initiated in the direction derived via the mean value of the tangential stress or the mean value of the strain energy density factor. Asymptotic expansion for the notch before the perturbation inception takes place reads

$$U^0(x) = H_1 r^{\delta_1} u_1(\theta) + H_2 r^{\delta_2} u_2(\theta) + \dots \tag{6}$$

where only singular terms are considered and r is the distance from the notch tip. The outer U^ε and inner expansion V^ε for the perturbed domain Ω^ε is

$$U^\varepsilon(x) = U^0(x) + k_1(\varepsilon) K_{1d(p)} r^{-\delta_1} u_{-1}(\theta) + k_2(\varepsilon) K_{2d(p)} r^{-\delta_2} u_{-2}(\theta) + \dots, \tag{7}$$

$$\begin{aligned} V^\varepsilon(y) = F_1(\varepsilon)\left[\rho^{\delta_1} u_1(\theta) + K_{1d(p)} \rho^{-\delta_1} u_{-1}(\theta) + K_{2d(p)} \rho^{-\delta_2} u_{-2}(\theta) + \dots\right] + \\ + F_2(\varepsilon)\left[\rho^{\delta_2} u_2(\theta) + K'_{1d(p)} \rho^{-\delta_1} u_{-1}(\theta) + K'_{2d(p)} \rho^{-\delta_2} u_{-2}(\theta) + \dots\right] + \dots . \end{aligned} \tag{8}$$

The terms in the brackets on the right-hand side of (8) are basis functions of inner expansion and their first terms describe their behavior for $\rho = r/\varepsilon \to \infty$, where ε is the small perturbation parameter. The $u_{-1}(\theta)$ and $u_{-2}(\theta)$, are dual (auxiliary) solutions to $u_1(\theta)$ and $u_2(\theta)$. The $K_{1d(p)}$, $K_{2d(p)}$ are calculated using the Ψ - integral in the inner domain whose remote boundary $\partial\Omega^{in}$ is subjected to the boundary condition $U\big|_{\partial\Omega^{in}} = \rho^{\delta_1} u_1(\theta)$

$$K_{1d(p)} = \frac{\Psi(V_1^h, \rho^{\delta_1} u_1)}{\Psi(\rho^{-\delta_1} u_{-1}, \rho^{\delta_1} u_1)}, \; K_{2d(p)} = \frac{\Psi(V_1^h, \rho^{\delta_2} u_2)}{\Psi(\rho^{-\delta_2} u_{-2}, \rho^{\delta_2} u_2)}, \tag{9}$$

where V_1^h is the finite element approximation to the first basis function of (8). Similarly, the coefficients $K'_{1d(p)}$, $K'_{2d(p)}$ are calculated in the inner domain whose remote boundary $\partial\Omega^{in}$ is subjected to the boundary condition $U\big|_{\partial\Omega^{in}} = \rho^{\delta_2} u_2(\theta)$

$$K'_{1d(p)} = \frac{\Psi(V_2^h, \rho^{\delta_1} u_1)}{\Psi(\rho^{-\delta_1} u_{-1}, \rho^{\delta_1} u_1)}, \; K'_{2d(p)} = \frac{\Psi(V_2^h, \rho^{\delta_2} u_2)}{\Psi(\rho^{-\delta_2} u_{-2}, \rho^{\delta_2} u_2)}, \tag{10}$$

where V_2^h is the finite element approximation to the second basis function of (8). The ratio of the debonding to the penetrating (propagating into material I or II) ERR is following

$$\frac{G_d}{G_p} = \frac{K_{1d}\Psi_1 + (K'_{1d}\Psi_1 + K_{2d}\Psi_2)\eta_d + K'_{2d}\Psi_2\eta_d^2}{K_{1p}\Psi_1 + (K'_{1p}\Psi_1 + K_{2p}\Psi_2)\eta_p + K'_{2p}\Psi_2\eta_p^2} \frac{a_d^{2\delta_1 - 1}}{a_p^{2\delta_1 - 1}}, \eta_{d,p} = \frac{H_2}{H_1} \frac{a_{d,p}^{\delta_2 - \delta_1}}{L^{\delta_2 - \delta_1}}, \tag{11}$$

$$\Psi_{1,2} \equiv \Psi(\rho^{-\delta_{1,2}} u_{-1,-2}, \rho^{\delta_{1,2}} u_{1,2}), \tag{12}$$

where $a_{d,p}$ is the length of the debonding or propagating crack, L is the characteristic length of the perturbed domain, Profant et al. (2011). The debonding occurs if the condition $G_d / G_p > G_c^i / G_c^{\mathrm{I,II}}$ is satisfied, where G_c^i is the interface toughness, $G_c^{\mathrm{I,II}}$ toughness of the material I, II.

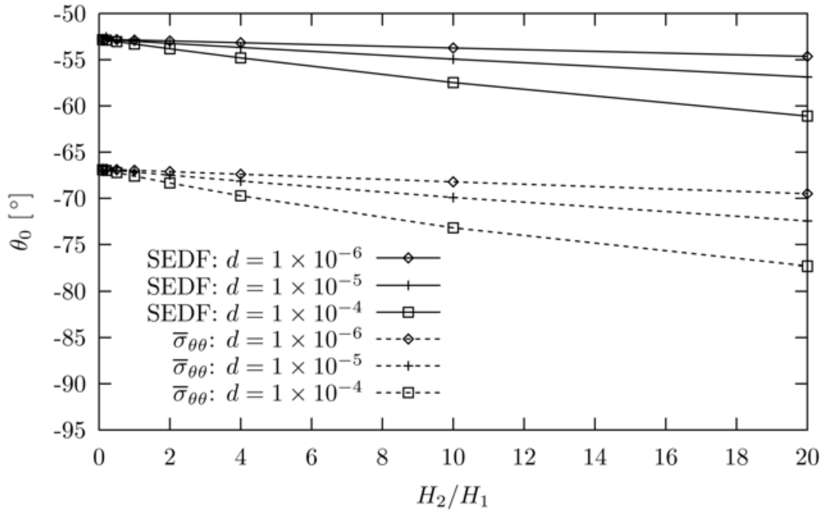

Fig. 2: Crack initiation angles for varying loading conditions (expressed by H_2/H_1) and for selected distances d.

3. Numerical results

The parametric study of the crack initiation directions is determined for specific geometry. The rectangular bi-material orthotropic notch characterized by angles $\omega_1 = 90°$ and $\omega_2 = 180°$ and materials with the Young's modulus $(E_x)_{\mathrm{I}} = 100$, $(E_y)_{\mathrm{I}} = 50$, $(E_x)_{\mathrm{II}} = 400$, $(E_y)_{\mathrm{II}} = 50$ [MPa] is considered. The corresponding stress singularity exponents have values $\delta_1 = 0.573$ and $\delta_2 = 0.941$. There are received GSIFs $H_1 = 6.43\,[MPa^{1-\delta_1}]$ and $H_1 = 39.99\,[MPa^{1-\delta_2}]$ under the remote loading $\sigma_{xx}^{appl} = 100$ [MPa] of the bi-material notch. The direction of the applied stress need not be specified directly, but it can be expressed by varying ratios $\Gamma_{21} = H_2 / H_1$. The angles θ_0 are determined on the basis of finding the maximum of the mean value of the tangential stress and the minimum of the strain energy density factor. For the considered material the results of the crack initiation direction are shown in Fig. 2. The averaging distance *d* was taken as 1e-4, 1e-5 and 1e-6. It is shown that the direction θ_0 depends on *d* especially for larger differences between H_1 and H_2.

4. Conclusions

It was developed the procedure for evaluation of the GSIFs of the bi-material notch using the Ψ - integral method and consequently it was developed the stability criterion of the notch. The matched asymptotic expansion method was introduced as the other stability criterion of the notch. Finally, the numerical study of the crack growth potential direction was shown.

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