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A THEORETICALLY CORRECT ALGORITHM FOR NONLINEAR CONSTITUTIVE MATRIX OF A SHELL

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Abstract: *A theoretically correct algorithm for nonlinear constitutive matrix of shell is introduced. The derivation starts with general formulas defining the constitutive matrices and it is applied to a specific problem of a shell respective to material nonlinearity.*

Keywords: ***Shell, constitutive matrix, material nonlinearity.***

1. Introduction

The paper starts from the basic relation defining a tensor of tangent material stiffness (e.g. Belytschko, Liu & Moran, 2000). From this definition a theoretically correct algorithm of the tangent constitutive matrix of a shell is derived.

2. Basic relations

Let us start from the relation for the tangent material stiffness (1), which can be applied for a wide scale of materials, where the material modulus $\mathbf{C}$ is the fourth order tensor, $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor and $\mathbf{E}$ is the Green-Lagrange strain tensor.

$$\mathbf{C}(\mathbf{E}) = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \tag{2}$$

When proceeding to the Voigt notation, we introduce the materal stiffness matrix $\overline{\mathbf{C}}$, the matrix of the second Piola-Kirchhoff stress $\overline{\mathbf{S}}$ and the matrix of the Green-Lagrange strain $\overline{\mathbf{E}}$. Then the equation (3) can be rewritten as follows:

$$\overline{\mathbf{C}}(\overline{\mathbf{E}}) = \frac{\partial \overline{\mathbf{S}}}{\partial \overline{\mathbf{E}}} \tag{4}$$

When a load increment is small enough then the constitutive relation (5) can be linearized.

$$\delta \overline{\mathbf{S}} = \overline{\mathbf{C}} \cdot \delta \overline{\mathbf{E}} \tag{6}$$

Then we can write the following relation for particular members of the constitutive matrix $\overline{\mathbf{C}}$:

$$\overline{C}_{ij} = \frac{\delta \overline{S}_i}{\delta \overline{E}_j} \tag{7}$$

With regard to linearity of the relations (8) and (9), for determination of ~~a~~ members of the constitutive matrix we can choose an arbitrary value of $\delta \overline{E}_j$, then also $\delta \overline{E}_j = 1$. Then we can easily determine members of the constitutive matrix $\overline{\mathbf{C}}$ as pertinent components of the stress vector $\overline{\mathbf{S}}$ for the unit magnitude of the strain vector $\overline{\mathbf{E}}$.

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$$\overline{C}_{ij} = \delta\overline{S}_i\left(\delta\overline{E}_j = 1\right)$$
(10)

The similar way can be used for obtaining members of the constitutive matrix of a shell. Let us define the vector of internal forces of a shell (11), where particular internal forces are defined in a usual way as integral factors of stress components (7).

$$\mathbf{S}^{(s)} = \begin{bmatrix} m_x & m_y & m_{xy} & v_x & v_y & n_x & n_y & n_{xy} \end{bmatrix}^{\mathrm{T}}$$
(12)

$$m_x = \int_h \sigma_x \, z \, \mathrm{d}z \qquad m_y = \int_h \sigma_y \, z \, \mathrm{d}z \qquad m_{xy} = \int_h \tau_{xy} \, z \, \mathrm{d}z$$
$$v_x = \int_h \tau_{xz} \, \mathrm{d}z \qquad v_y = \int_h \tau_{yz} \, \mathrm{d}z$$
$$n_x = \int_h \sigma_x \, \mathrm{d}z \qquad n_y = \int_h \sigma_y \, \mathrm{d}z \qquad n_{xy} = \int_h \tau_{xy} \, \mathrm{d}z$$
(13)

Let us define the strain vector of a shell in a usual way.

$$\mathbf{E}^{(s)} = \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} & \gamma_{xz} & \gamma_{yz} & \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^{\mathrm{T}}$$
(14)

$$\kappa_x = \frac{\partial \varphi_y}{\partial x} \qquad \kappa_y = -\frac{\partial \varphi_x}{\partial y} \qquad \kappa_{xy} = \frac{\partial \varphi_y}{\partial y} - \frac{\partial \varphi_x}{\partial x}$$
$$\gamma_{xz} = \frac{\partial u_z}{\partial x} + \varphi_y \qquad \gamma_{yz} = \frac{\partial u_z}{\partial y} - \varphi_x$$
$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \varepsilon_y = \frac{\partial u_y}{\partial y} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Similar relation as the equation (15) can be written also for the constitutive matrix of a shell:

$$\mathbf{C}^{(s)} = \frac{\partial \mathbf{S}^{(s)}}{\partial \mathbf{E}^{(s)}}$$
(16)

To obtain particular members of the constitutive matrix of a shell, similar relation as in the equation (17) can be written:

$$C_{ij}^{(s)} = \frac{\delta S_i^{(s)}}{\delta E_j^{(s)}} \tag{18}$$

With regard to linearity of the constitutive matrix in each iteration step, particular members of the constitutive matrix can be again determined as the pertinent components of the vector $\delta S_i^{(s)}$ for unit value of the strain component $\delta E_j^{(s)}$.

3. Algorithm of the calculation of the constitutive matrix of a shell

3.1. Layered shell element

Inasmuch as the internal forces $\delta S_i^{(s)}$ corresponding to the strain $\delta\overline{E}_j$ must be obtained by numerical integration (Šolín, Segeth & Doležel 2004), the shell must be didvided along its thickness h into layers. A layer i is determined by its thickness $h_{lr,i}$ and by the location of its central surface $z_{lr,i}$. The pertinent integrals can be evaluated by Gauss quadrature formula which defines the location of

Gaussian points $z_{gp,j}$. This quadrature formula gives exact results for polynomials of the $(2n-1)$-th and lower order, where n is the number of Gaussian points in each layer.

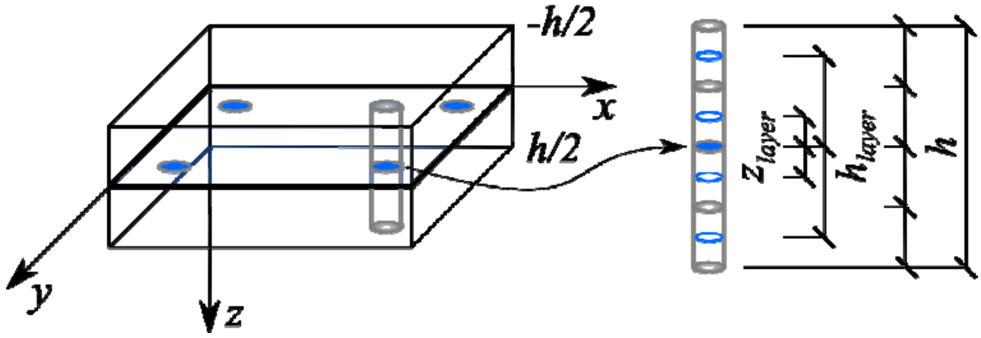

Fig. 1: Division of element along its thickness h into 4 layers with one Gaussian point in each layer $z_{lr,i} = z_{gp,j}$.

3.2. Bending and membrane members of the constitutive matrix

A bending and membrane members of the constitutive matrix of a shell $\mathbf{C}^{(s)}$ are calculated from the constitutive matrices of layers $\mathbf{c}_{lr,i}$ (11) transformed into such coordinate system in which the shell constitutive matrix $\mathbf{C}^{(s)}$ should be assembled.

$$\mathbf{c}_{lr,i} = \left(\mathbf{T}_c^{-1}\right)^{\mathrm{T}} \mathbf{c}_{lr,i}^{local} \mathbf{T}_c^{-1} = \begin{bmatrix} c_{lr,i,xxxx} & c_{lr,i,xxyy} & c_{lr,i,xxxy} \\ c_{lr,i,yyxx} & c_{lr,i,yyyy} & c_{lr,i,yyxy} \\ c_{lr,i,xyxx} & c_{lr,i,xyyy} & c_{lr,i,xyxy} \end{bmatrix} \tag{19}$$

For assemblage of the constitutive matrix $\mathbf{C}^{(s)}$ the equation (12) shall be used. When chosing the first member κ_x of the deformation vector $\mathbf{E}^{(s)}$ equal to one, and the remaining members of this vector are zero, then the vector of internal forces $\mathbf{S}^{(s)}$ is equal to the first column of the constitutive matrix $\mathbf{C}^{(s)}$.

$$\mathbf{S}^{(s)} = \mathbf{C}^{(s)} \mathbf{E}^{(s)} \tag{20}$$

$$\begin{bmatrix} m_x \\ m_y \\ m_{xy} \\ v_x \\ v_y \\ n_x \\ n_y \\ n_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{61} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{71} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \kappa_x = 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} C_{11} = m_x \\ C_{21} = m_y \\ C_{31} = m_{xy} \\ C_{41} = v_x \\ C_{51} = v_y \\ C_{61} = n_x \\ C_{71} = n_y \\ C_{81} = n_{xy} \end{matrix}$$

This algorithm will be used for evaluating the first three and the last three columns of the constitutive matrix $\mathbf{C}^{(s)}$. The chosen vector of deformation $\mathbf{E}^{(s)}$ containing only one nonzero member i.e. curvature $\kappa_x = 1$ will yield the strain in the layers as follows.

$$\varepsilon_{lr,i,x} = \varepsilon_x + \kappa_x z_{lr,i} \qquad \varepsilon_{lr,i,y} = \varepsilon_y + \kappa_y z_{lr,i} \qquad \gamma_{lr,i,xy} = \gamma_{xy} + \kappa_{xy} z_{lr,i} \tag{21}$$

A constitutive matrix of a layer $\mathbf{c}_{lr,i}$ obtained from a nonlinear calculation will be multiplied by the strain vector $\boldsymbol{\varepsilon}_{lr,i}$ to obtain the pertinent stress vector.

$$\boldsymbol{\sigma}_{lr,i} = \mathbf{c}_{lr,i} \boldsymbol{\varepsilon}_{lr,i} \tag{22}$$

Then the stress $\boldsymbol{\sigma}_{vr,i}$ in each layer will be integrated related to the central surface of the shell by the realtion (7). The resulting vector of the internal forces $\mathbf{S}^{(s)}$ will be substituted into the first column of the constitutive matrix $\mathbf{C}^{(s)}$. This procedure will be repeated also for the remaining columns of the constitutive matrix except the fourth a fifth one, which will be evaluated by a different procedure.

3.3. Shear members of the shell constitutive matrix

To complete all the members of the constitutive matrix $\mathbf{C}^{(s)}$ it remains to determine the shear stiffnesses C_{44} in the x direction and C_{55} in the direction.The C_{44} and C_{55} stiffnesses will be calculated by the relations (23) that were derived from the demand of the equivalence of the virtual work of the 3D and the 2D models.

$$C_{44} = \frac{1}{\displaystyle\int_h \frac{1}{G_{lr,i,x}} \left[\frac{\int_z^h E_{lr,i,x}\, \overline{z}_{lr,i}\, \mathrm{d}\overline{z}}{\int_h E_{lr,i,x}\, \overline{z}^2_{lr,i}\, \mathrm{d}\overline{z}} \right]^2 \mathrm{d}z} \qquad C_{55} = \frac{1}{\displaystyle\int_h \frac{1}{G_{lr,i,y}} \left[\frac{\int_z^h E_{lr,i,y}\, \overline{z}_{lr,i}\, \mathrm{d}\overline{z}}{\int_h E_{lr,i,y}\, \overline{z}^2_{lr,i}\, \mathrm{d}\overline{z}} \right]^2 \mathrm{d}z} \tag{24}$$

$E_{lr,i,x}$, $E_{lr,i,y}$, $G_{lr,i,x}$ and $G_{lr,i,y}$ are the Young and shear modules of each layer.

4. Conclusions

The paper has shown a theoretically correct and practically useful algorithm for calculation of tangent constitutive matrix of a shell. This algorithm is applied in the RFEM program for finite element analysis of structures (Němec et al., 2010).

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