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COMPARISON OF P1P1 AND Q1P0 "TAYLOR-HOOD ELEMENTS" IN FLOW PROBLEMS

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Abstract: *This paper deals with comparison of two low-order finite elements used to describe homogenous incompressible flow in two dimensions with Eulerian description. The governing equations are presented as well as their discretized form obtained using traditional Galerkin method. Necessary stabilization techniques are discussed, allowing using interpolations violating LBB condition, application to convection-dominated problems, etc. The elements are compared on an example of lid cavity driven flow.*

Keywords: ***Lid-driven, cavity, flow, stabilization, level set.***

1. Introduction

This paper is focused on comparison of different low-order elements used to model homogenous fluid flows. In the longer perspective, these elements will be used to simulate casting of fresh concrete. In present approach, the fluid is considered as homogenous continuous medium, the flow is described by Navier-Stokes equations and solved by the means of Finite element method. There are in principle three ways, how to describe motion of continuous medium. In Lagrangian description, motion of each point is described in framework of reference configuration. This approach is usually used in structural mechanics, for fluid dynamics is suitable only if discrete particle model is used. Otherwise, large deformations requires frequent remeshing. In Eulerian description, motion is connected to actual configuration and therefore, convective term is present. In this case, computation can be done on a fixed grid and no remeshing is needed. So called ALE formulation combines both and is proper in fluid-structure interaction. Further information can be found for example in (Donea & Huerta, 2003). In the present work, Eulerian formulation is used. Due to the presence of convective terms, an additional stabilization is needed. Two types of elements are used: Q1P0, a quadrilateral element with linear approximation of velocity and constant approximation of pressure and T1P1, a triangular element with linear approximation for both velocity and pressure fields. Either T1P1 or Q1P0 do not satisfy so called LBB stability condition, so proper pressure stabilization technique is necessary.

2. Governing equations, weak formulation and discretization

The flow of continuous Newtonian fluid is described by Navier-Stokes equation (1), which represents balance of momentum. Further, an incompressibility is assumed and therefore the velocity field must be divergence free (2):

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho\, \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{b} \quad , \quad (\boldsymbol{x}, t) \in \Omega_t \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \;\; , \;\; (\boldsymbol{x}, t) \in \Omega_t \, . \tag{2}$$

Standard Dirichlet (3) and von Neumann (4) boundary condition are prescribed at the complementary parts of domain boundary:

$$\boldsymbol{u} = \boldsymbol{g} \;\; , \; \boldsymbol{x} \in \Gamma_{\boldsymbol{g}} \tag{3}$$

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$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{h}\, ,\, \boldsymbol{x} \in \Gamma_h \tag{4}$$

In equations (1-4), $\boldsymbol{\sigma}$ is stress tensor, $\boldsymbol{u}$ is velocity field, $\boldsymbol{b}$ vector of body forces and ρ is fluid density.

Weak formulation of problem (1-4) can be obtained using traditional Galerkin method. However, the standard Galerkin method is not very suitable for convection-dominated problems. In such case, non-physical oscillations are generated and computed solution is not realistic, as it is reported for example in (Donea & Huerta, 2003). Therefore, some stabilization technique to prevent the oscillations is needed. In this work, SUPG (Streamline Upwind – Petrov/Galerkin) and LSIC (Least squares on incompressibility constraint) stabilizations are used, but also different techniques are possible, for further references, see (Tezduyar, 2000), for example.

Another problem arising from numerical computations is due to treating incompressibility condition. Due to this assumption is not possible to compute pressure from any constitutive equation; pressure is another degree of freedom. In fact, pressure is a Lagrange multiplier on incompressibility constraint, and thus is determined by satisfying incompressibility condition. Function spaces for velocity and pressure are not independent, sufficient condition for convergence is so called LBB (Ladyzenska-Babuska-Brezzi) condition. This condition can be broken and realistic solutions can be obtained, but some other stabilization, for example PSPG (Pressure stabilizing-Petrov Galerkin) is needed.

After discretization, provided that proper finite element spaces are defined, stabilized finite element formulation of problem (1-4) can be stated as follows: find $\boldsymbol{u}^h \in \boldsymbol{S}_u^{\,h}$ and $p^h \in \boldsymbol{S}_p^{\,h}$ such that $\forall\ \boldsymbol{w}^h \in \boldsymbol{V}_u^{\,h}$ and $\forall\ q^h \in \boldsymbol{V}_p^{\,h}$ holds

$$\begin{aligned}
&\int_\Omega \rho\, \boldsymbol{w}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial t}\, d\Omega + \int_\Omega \rho\, \boldsymbol{w}^{\mathbf{h}} \cdot \left(\boldsymbol{u}^h \cdot \nabla\, \boldsymbol{u}^h\right) d\Omega + \int_\Omega \nabla\, \boldsymbol{w}^{\mathbf{h}} : \boldsymbol{\tau}\left(\boldsymbol{u}^h\right) d\Omega - \int_\Omega \boldsymbol{w}^{\mathbf{h}} \cdot \nabla p^h\, d\Omega \\
&\quad - \int_\Omega \boldsymbol{w}^{\mathbf{h}} \cdot \boldsymbol{b}\, d\Omega - \int_\Gamma \boldsymbol{w}^h \cdot \boldsymbol{h}\, d\Gamma + \int_\Omega q^h \left(\nabla \cdot \boldsymbol{u}^h\right) d\Omega \\
&\quad + \sum_{el} \left[\int_{\Omega_e} \tau_{SUPG} \left(\boldsymbol{u}^h \cdot \nabla\, \boldsymbol{w}^h\right) \cdot \left(\rho \frac{\partial \boldsymbol{u}^h}{\partial t} + \rho\, \boldsymbol{u}^h \cdot \nabla\, \boldsymbol{u}^h - \nabla \cdot \boldsymbol{\tau}\left(\boldsymbol{u}^h\right) + \nabla p^h - \boldsymbol{b} \right) d\Omega_e \right] \\
&\quad + \sum_{el} \left[\int_{\Omega_e} \tau_{PSPG} \frac{1}{\rho} \nabla q^h \cdot \left(\rho \frac{\partial \boldsymbol{u}^h}{\partial t} + \rho\, \boldsymbol{u}^h \cdot \nabla\, \boldsymbol{u}^h - \nabla \cdot \boldsymbol{\tau}\left(\boldsymbol{u}^h\right) + \nabla p^h - \boldsymbol{b} \right) d\Omega_e \right] \\
&\quad + \sum_{el} \left[\int_{\Omega_e} \tau_{LSIC} \nabla \cdot \boldsymbol{w}^{\mathbf{h}}\, \rho \nabla \cdot \boldsymbol{u}^{\mathbf{h}}\, d\Omega_e \right] = 0
\end{aligned} \tag{5}$$

Terms in the first two lines follow from standard Galerkin discretization, terms in the third line are due to SUPG stabilization, because of convective effects, terms in the fourth line provide PSPG stabilization (LBB condition is not satisfied), and last term provides additional stability for high velocities. Coefficients τ_{SUPG}, τ_{PSPG} and τ_{LSIC} can be computed as norm of certain terms of (5), as it is done in present work for Q1P0 element. In the case of T1P1 element, coefficients are computed in different way, using so called UGN-stabilization, which is based on characteristic element length, see (Tezduyar, 2000).

3. Numerical results

The performance of studied elements has been compared using lid driven cavity test, which is often used as a benchmark. The results are presented for solution corresponding to Reynolds number RE=100. In case of T1P1 element, results has been obtained at first as a solution to Stokes problem, where time dependent term is omitted and later also as a solution of a full Navier-Stokes problem. Used grid has 400 dofs and 722 elements. The results shown in Fig. 1 compare the velocity profiles along vertical line at the center of specimen. The response of both elements is compared to results obtained by T2P1 Taylor-Hood element with quadratic approximation of velocity and constant pressure (on the same grid, so that the number of velocity degrees of freedom is doubled), and reference solution from (Marchi at al., 2009) obtained on a fine grid of 1024x1024 nodes. The pressure profiles are shown in Fig. 2, where the colors distinguish solutions obtained using different elements (red-Q1P0, green-T1P1 elements). The raw pressure profile for Q1P0 element exhibits spurious oscillations, but the post-processed profile (as shown in Fig. 2) shows a good agreement with other solutions.

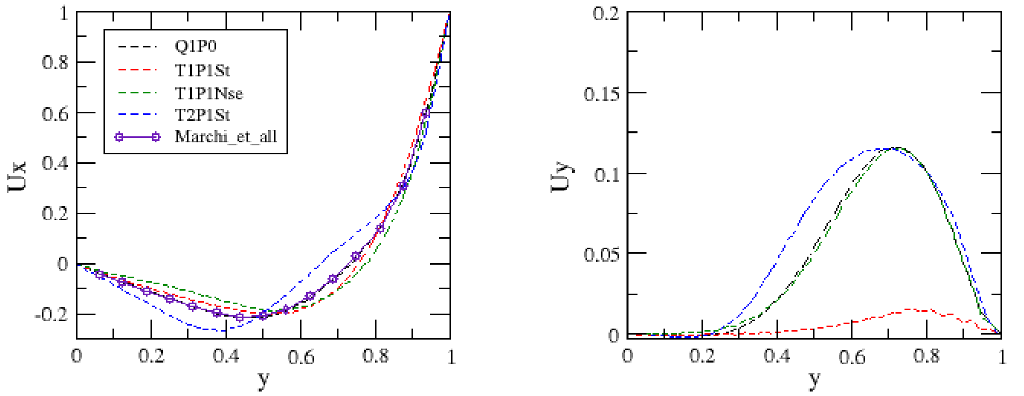

Fig. 1: Lid-driven cavity: velocity profiles along vertical line at specimen center for Re=100.

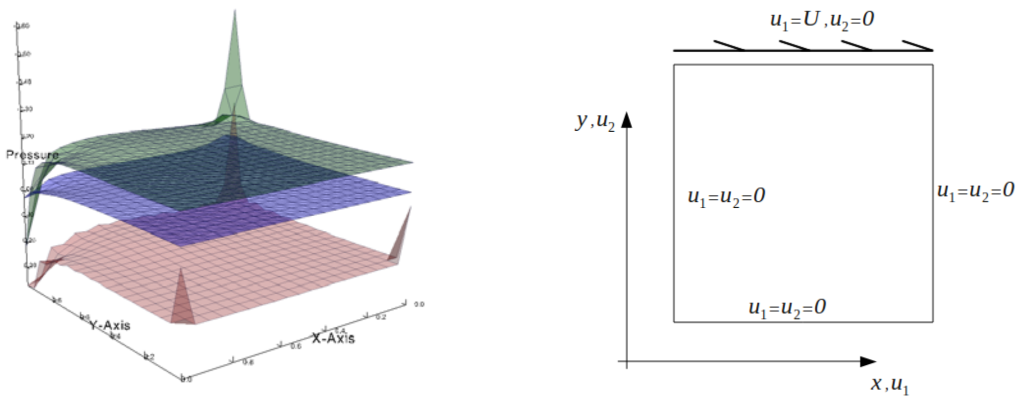

Fig. 2: Lid driven cavity: pressure profiles and lid driven test scheme.

4. Level set method

In a longer perspective, results from this work will be used for modeling of flow with free surface. If governing equations of motion are formulated in Eulerian sense and computations are done on fixed domain (mesh), an interface-tracking technique is needed, see (Tezduyar 2006). In this work technique based on level set method is used. Main idea of this method is to represent interface as a zero level set of a suitable higher-dimensional function. Interface is then manipulated implicitly through this so called level set function. For further reference, see (Osher & Fedkiw 2001). The level set function is typically defined as a signed distance function of points from the interface. Governing equation for interface moving by convective velocity $\boldsymbol{u}$ can be written as:

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \nabla \phi = 0 \tag{6}$$

This in fact says, that total derivative of ϕ is equal to zero. Equation (6) can be reformulated into form of Hamilton-Jacobi equation, which can be solved using positive explicit scheme. Drawback of level set method is that is not conservative, when assumption of divergence free velocity is used. Therefore, one has to use some re-initialization technique to recover sign distance property of level set function. This can be found for example in (Osher & Fedkiw 2003). The initial results are shown in Fig. 3, where squared domain is transported through the mesh, due to prescribed velocity, by convective effects.

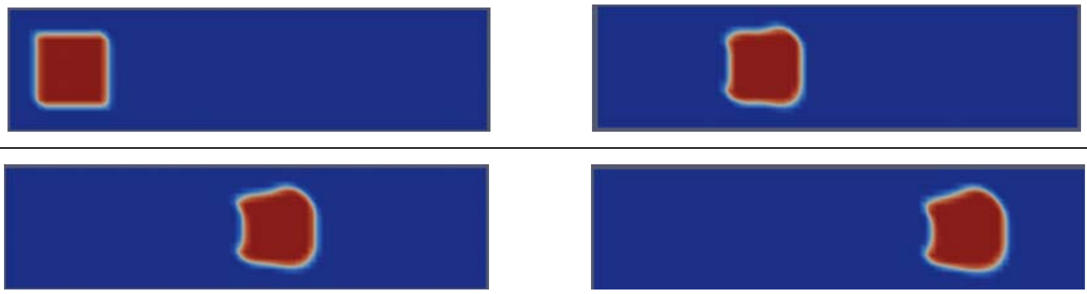

Fig. 3: Level set method: Square moving along mesh by convective velocity.

5. Conclusions

To conclude, satisfying agreement with both T1P1 and Q1P0 elements has been obtained. Reasons for differences with T2P1 element is to be further analyzed, but partially can be explained by better approximation properties of T2P1 and by the fact that analysis with T2P1 elements was done with number of DOFs almost doubled. In problems with interface-tracking, initial results are promising, however deformations of transported domain can be observed. This can attributed to poor approximation of T1P0 and to the coarse mesh used.

Acknowledgement

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