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STRESS SINGULARITY ANALYSIS OF CRACKS LYING ON THE INTERFACE BETWEEN TWO ORTHOTROPIC MATERIALS

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Abstract: *This article deal with problems of cracks in bodies with bimaterial interface. The stress and displacement near the front of crack located at the interface between two orthotropic materials is analyzed both analytically and numerically using FEM. Analytical description of the singular stress field and displacements is obtained using the anisotropic generalized complex potentials of plane elasticity.*

Keywords: ***Orthotropic bimaterials, interface crack, conservative integrals, stress intensity.***

1. Introduction

The problem of a crack propagating along an interface between two materials is of great importance to industry. With increasing application of composite materials consisting of long fibre reinforced composite layers in different directions there is growing interest into the understanding of interface fracture between two orthotropic materials. To this end, accurate and efficient methods are required for determining the stress and displacement fields in the neighbourhood of crack tip. Before proceeding, relevant concepts related to interface cracks in orthotropic bimaterials are presented.

2. Fracture mechanics of interface cracks between two orthotropic materials in plane strain

Plane strain deformation can be treated by a change of compliances of an orthotropic material s_{ij} as

$$s'_{ij} = s_{ij} - \frac{s_{i3}s_{3j}}{s_{33}}, \quad i,j = 1,2,6\,. \tag{1}$$

Using the newly defined material parameter $s_{\pm} = \sqrt{\sqrt{s'_{11}s'_{22}} \pm \left(s'_{12} + s'_{66}/2\right)}$, the real Barnett-Lothe tensors **L** and **S** can be expressed following Ting (1996) such as

$$\mathbf{L}^{-1} = \sqrt{2}s + \begin{bmatrix} \sqrt{s'_{11}} & 0 \\ 0 & \sqrt{s'_{22}} \end{bmatrix}, \ \mathbf{SL}^{-1} = \left(\sqrt{s'_{11}s'_{22}} + s'_{12}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

Further, it is convenient to introduce a symmetric, positive definite matrix **D** and a symmetric matrix **W** defined by

$$\mathbf{D} = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}, \ \mathbf{W} = \begin{bmatrix} + & -w \\ w & 0 \end{bmatrix} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}\,, \tag{3}$$

where subscripts 1 and 2 refer to the material 1 and 2 respectively.

The in-plane stresses in the neighbourhood of a crack tip at an interface are given by

$$\sigma_{ij} = \frac{1}{\sqrt{2\pi r}}\left[\mathrm{Re}\left(Kr^{i\varepsilon}\right)\Sigma_{ij}^{(1)}(\theta,\varepsilon) + \mathrm{Im}\left(Kr^{i\varepsilon}\right)\Sigma_{ij}^{(\hat{e})}(\theta,\varepsilon)\right], \tag{4}$$

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where $i,j=1,2$, the complex stress intensity factor $K=K_1+iK_2$ and the superscript (1) and (2) are related to the real and imaginary parts of $Kr^{i\varepsilon}$. ε is referred to as the oscillatory index defined by

$$\varepsilon = 1/2\pi \ln\left(1+\beta/1-\beta\right), \tag{5}$$

where β is the generalised Dundurs parameter $\beta = -w/\sqrt{D_{11}D_{22}}$. The complex stress intensity factor K in Eq. (4) exhibits three principal drawbacks–it contains the logarithm of length, its physical dimension depends on ε and the phase angle arg(K) depends on the used length unit. To remove these drawbacks a reference length l is usually introduced which defines the new complex stress intensity factor $\widehat{K} = Kl^{i\varepsilon}$.The subsequent description of the singular elastic field employs the dimensionless matrix $R(c)$ which is defined as a function of complex number c (see Mantic (2004))

$$\mathbf{R}[c] - \mathrm{Re}[c]\mathbf{I} + \beta^{-1}\,\mathrm{Im}[c]\mathbf{D}^{-1}\mathbf{W} = \begin{bmatrix} \mathrm{Re}[c] & \mathrm{Im}[c]\sqrt{D_{22}/D_{11}} \\ -\mathrm{Im}[c]\sqrt{D_{11}/D_{22}} & \mathrm{Re}[c] \end{bmatrix} \tag{6}$$

and the traction ahead of crack along the interface are expressed for r→0 as:

$$\begin{bmatrix} \sigma_{xy} \\ \sigma_{yy} \end{bmatrix}(r,0) = \begin{bmatrix} \sigma_{xy}^{sing} \\ \sigma_{yy}^{sing} \end{bmatrix}(r,0) + O(1) = \frac{1}{\sqrt{2\pi r}}\mathbf{R}\left[\left(\frac{r}{l}\right)^{i\varepsilon}\right]\widehat{\mathbf{K}} + O(1),\ \text{for } r \to 0,\quad \widehat{\mathbf{K}} = \begin{bmatrix} \widehat{K}_2 \\ \widehat{K}_1 \end{bmatrix}, \tag{7}$$

while the crack face displacements in the vicinity of the crack tip are found to be

$$\begin{bmatrix} \Delta u_x \\ \Delta u_y \end{bmatrix}(r) = \begin{bmatrix} \Delta u_x^{sing} \\ \Delta u_y^{sing} \end{bmatrix}(r) + O(1) = \sqrt{\frac{r}{2\pi}}\,\frac{2\mathbf{D}}{\cosh \pi\varepsilon}\mathbf{R}\left[\frac{1}{1+2i\varepsilon}\left(\frac{r}{l}\right)^{i\varepsilon}\right]\widehat{\mathbf{K}} + O(r), \quad r \to 0\,. \tag{8}$$

The phase angle of mode mixity ψ is defined as follows:

$$\psi = \arg\left(\widehat{K}_1 + i\sqrt{D_{11}/D_{22}}\,\widehat{K}_2\right) = \arg\left(\sqrt{D_{11}/D_{22}}\,\sigma_{xy}^{sing}/\sigma_{yy}^{sing}(l,0)\right). \tag{9}$$

3. Conservative integral

The reciprocal theorem of elastostatics states that in the absence of body forces and residual stresses the following integral is path independent

$$\Psi(\mathbf{u},\mathbf{u}') = \int_{\Gamma}\left(\sigma_{ij}(\mathbf{u})\cdot n_i u'_j - \sigma_{ij}(\mathbf{u}')\cdot n_i u_j\right)ds\,, \tag{10}$$

where Γ is any contour surrounding the crack tip and **u**, **u’** are two admissible displacement fields and $\sigma_{ij}(\mathbf{u})$, $\sigma_{ij}(\mathbf{u}')$ are the corresponding stress fields. Hence

$$\int_{\Gamma_1}\left(\sigma_{ij}(\mathbf{u})\cdot n_i u'_j - \sigma_{ij}(\mathbf{u}')\cdot n_i u_j\right)ds = \int_{\Gamma_2}\left(\sigma_{ij}(\mathbf{u})\cdot n_i u'_j - \sigma_{ij}(\mathbf{u}')\cdot n_i u_j\right)ds\,. \tag{11}$$

For the path far from the crack tip, the unprimed field is obtained from solution to the problem numerically by FE, for example. Along the vanishingly small path, the unprimed field is the singular eigensolution to the crack problem given in Eq.(7) and (8). The primed solution is chosen so that this integral identically yields the sought stress intensity factor. The primed solution (the auxiliary solution) corresponds to the singular eigensolution with the eigenvalue -1/2-iε for the displacement field. It has unbounded energy near the crack tip and thus corresponds to some concentrated source at the crack tip. It is a mathematical tool which allows extracting asymptotic coefficient terms from the complete exact solution. The generic form of the auxiliary traction vector **t’** acting along the contour Γ was derived as follows

$$\mathbf{t}'(z) = \left(\mathbf{I} + \overline{\mathbf{H}}^{-1}\mathbf{H}\right)\frac{\mathrm{e}^{-\pi\varepsilon}\,k_1\mathbf{w}'z^{-\mathrm{i}\varepsilon} + \mathrm{e}^{\pi\varepsilon}\,\overline{k_1\mathbf{w}'}z^{\mathrm{i}\varepsilon}}{z^{\frac{3}{2}}\,\mathrm{e}^{-\pi\varepsilon}} \tag{12}$$

where **I** stands for unit matrix, $z = x + \mu y$, where μ is a complex characteristic material number with positive imaginary part, **w**' is the complex eigenvector, k_1 is a complex constant, and **H** is the bimaterial matrix for two orthotropic materials with aligned principal axes

$$\mathbf{H} = \begin{bmatrix} H_{11} & -i\beta\left(H_{11}H_{22}\right)^{1/2} \\ i\beta\left(H_{11}H_{22}\right)^{1/2} & H_{22} \end{bmatrix} \tag{13}$$

whose components can be found in Suo (1990).

4. Numerical results

Bimaterial orthotropic strip subjected to tensile loading along the boundary parallel to the interface with an edge interface crack was modelled by FE with the system ANSYS. The material properties are listed in the Tab. 1. The phase angle of mode mixity ψ_k was calculated from Eq. (9) for l = 1 mm providing the value ψ_k = -0.178 rad. The angular dependence of analytically obtained singular field along a circular contour surrounding the crack tip was compared with the numerically obtained field at r = 1 mm using a very fine mesh.

Tab. 1: Elastic properties of the orthotropic bimaterial strip and corresponding values of Dundurs' parameters α, β.

	E_L [GPa]	E_T [GPa]	E_Z [GPa]	ν_{TZ}	ν_{ZL}	ν_{TL}	G_{TZ} [GPa]	G_{ZL} [GPa]	G_{TL} [GPa]	α	β
Mat 1	200	80	80	0.3	0.3	0.3	30	30	30	- 0.1193	0.1024
Mat 2	50	150	150	0.3	0.3	0.3	30	30	30		

The angular parts of the asymptotic stress-displacement field are plotted as functions of the polar angle θ for several values of the phase angle of mode mixity in Figs. 1 - 5.

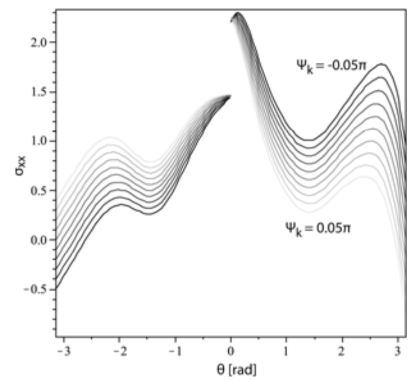

Fig. 1: Asymptotic stress σ_{xx} along the circular path for several values of the phase angle ψ_k.

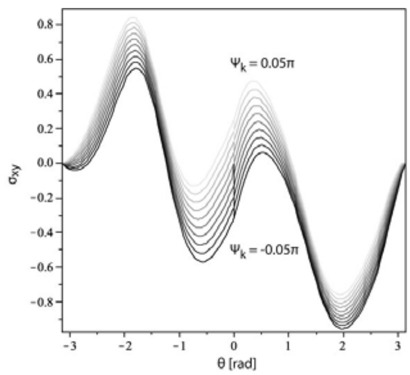

Fig. 2: Asymptotic stress σ_{xy} along the circular path for several values of the phase angle ψ_k.

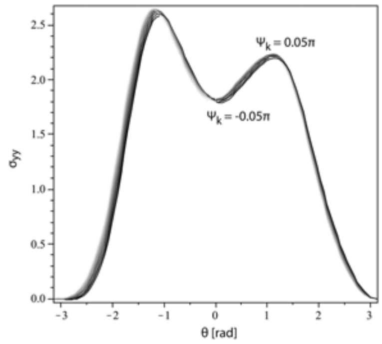

Fig. 3: Asymptotic stress σ_{yy} along the circular path for several values of the phase angle ψ_k.

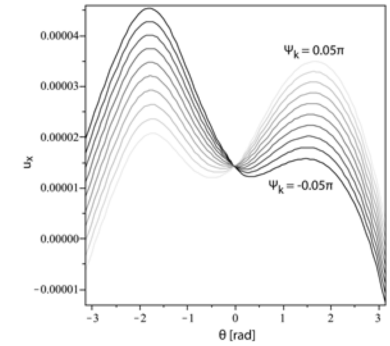

Fig. 4: Asymptotic displacement u_x for several values of the phase angle ψ_k.

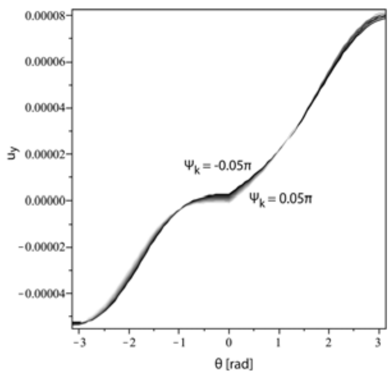

Fig. 5: Asymptotic displacement u_y for several values of the phase angle ψ_k.

It is clearly seen that the stress components σ_{yy} and the displacement components u_y depend only very little on the phase angle.

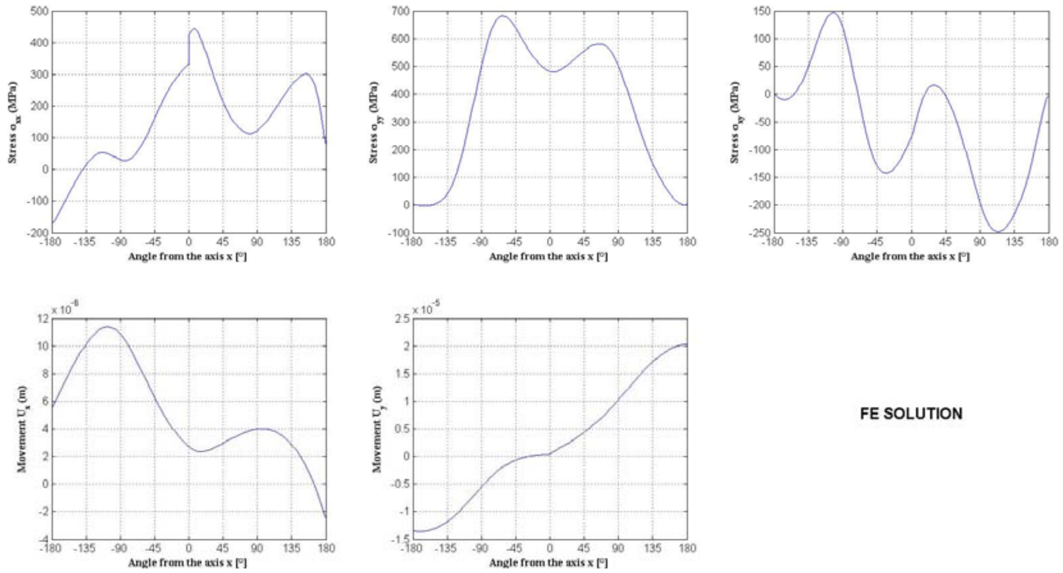

Fig. 6: FE solution to the stress-displacement field near the crack tip.

5. Conclusion

The stress field and the displacement field, calculated by FEM at the distance r = 1 mm from the crack tip is shown in Fig. 6. Observe that angular dependences of all quantities perfectly match with the analytical singular solutions. It means that the domain of dominance of the singular solution is of order of 1 mm for the examined specimen/crack configuration. The auxiliary solution defined in Eq. (12) was also evaluated and substituted into the conservative integral in Eq. (10), which enables us to calculate the real and imaginary part of the complex stress intensity factor. Further results will be presented at the conference.

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