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USAGE OF NONLINEAR NORMAL MODES IN DYNAMICS

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Abstract: *The paper summarizes the method of nonlinear normal modes (NNMs) which has gained importance in nonlinear dynamics. Two main definitions of NNMs are discussed: Rosenberg's definition and Shaw and Pierre definition based on geometric arguments and inspired by the centre manifold technique. Fundamental properties of NNMs like frequency-energy dependence, modal interactions and mode bifurcations and stability are introduced. To compute the NNMs analytical and numerical approaches are used and compared. The NNMs approach to nonlinear dynamics is clearly demonstrated and compared with linear normal modes (LNMs) using a smooth nonlinear mechanical system.*

Keywords: ***Nonlinear normal modes, mode bifurcation, stability, modal analysis.***

1. Introduction

The concept of normal modes is central in the theory of linear vibrating systems. The linear normal modes (LNMs) have interesting mathematical properties. They can be used to decouple the governing equations of motion, where a linear system vibrates as if it were made of independent oscillators governed by the eigensolutions. Moreover, free and forced oscillations can be expressed as linear combination of individual LNM motions. LNMs are relevant dynamical features that can be exploited for various purposes.

In real-life applications, nonlinearity is a frequent occurrence. Typical nonlinearities include backlash and friction in control surface, hardening nonlinearities, saturation effects, etc. Further, for instance, structural behaviour of materials made of composites is deviating significantly from linearity too. Any attempt to apply traditional linear analysis to nonlinear systems results at best in suboptimal design. In this context, NNMs offer a solid theoretical and mathematical tool for interpreting wide class of nonlinear dynamical phenomena, yet they have a clear and simple conceptual relation to the LNMs. Moreover, they can be advantageously used for the reduction of nonlinear models (Gabale & Sinha, 2011). The objective of the present paper is to describe and illustrate in a simple manner fundamental properties of NNMs for conservative nonlinear systems.

2. Basic definitions of NNMs

There exist two main definitions of the NNMs in the literature, due to Rosenberg and Shaw and Pierre (Kerschen et al., 2009). Historically, Lyapunov's and Poincaré's contributions served as the cornerstone of the NNMs development. Lyapunov showed that there exist at least n different families of periodic solutions around the stable equilibrium point of n-DOF conservative systems with no internal resonances. At low energy, the periodic solutions of each family are in the neighbourhood of a LNM of corresponding linearized system. These n families define n NNMs that can be regarded as nonlinear extensions of the n LNMs of the corresponding linear system (Vakakis et al., 1996).

During the normal mode motion of a linear conservative system, each system component moves with the same frequency and with a fixed ratio amongst displacements of the components. Targeting a straightforward nonlinear extension of the LNM concept, Rosenberg defined a NNM as a *vibration in unison* of the system (i.e. synchronous oscillation). This definition requires that all points of the system reach their extreme values and pass through zero simultaneously and allows all displacements to be expressed in terms of a single reference displacement.

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Shaw and Pierre proposed a generalization of Rosenberg's definition that provides a direct extension of the NNMs concept to damped systems. Based on geometric arguments and inspired by the center manifold technique, they defined a NNM as two dimensional invariant manifold in phase space. Such a manifold is invariant under the flow, which extends the invariance property of LNMs to nonlinear systems (Shaw & Pierre, 1993).

3. Fundamental properties of NNMs

NNMs have intrinsic properties that are fundamentally different from those of LNMs. They are reviewed and some of them are illustrated in this chapter.

3.1. Frequency-energy dependence

One typical dynamical feature of nonlinear systems is the *frequency-energy dependence* of their oscillations. Let us consider a 2-DOF conservative system with a cubic stiffness, which is governed by the equations

$$\begin{aligned} \ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 &= 0, \\ \ddot{q}_2 + (2q_2 - q_1) &= 0. \end{aligned} \tag{1}$$

To solve the system (1), harmonic balance method can be applied. This method expresses the periodic motion of a system by means of finite Fourier series (Nayfeh, 1996). The solution can be expressed as two cosine functions with different amplitudes for each coordinate, respectively. Substituting the solution into (1), one can obtain the unknown amplitudes in dependence on frequency of the periodic motion. Based on this, total energy of the system can be calculated. Due to the frequency-energy dependence, the representation of NNMs in a frequency-energy plot (FEP) is used (Peeters et al., 2009). A NNM motion is represented by a point in the FEP which is drawn at a frequency corresponding to the minimal period of the periodic motion and at energy equal to the conserved total energy during the motion. Each branch represents a family of NNM motion with the same qualitative features (Fig. 1).

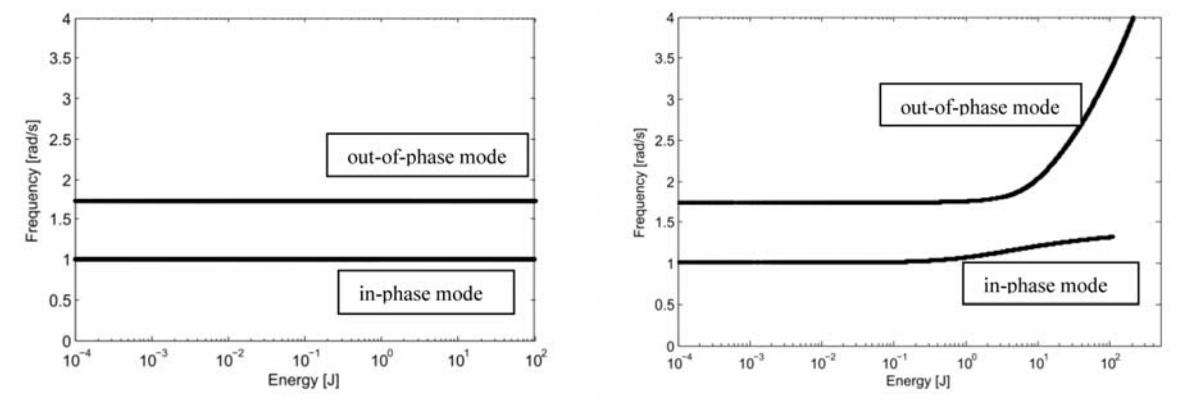

Fig. 1: Frequency-energy plot of system (1), (left – LNMs, right – NNMs).

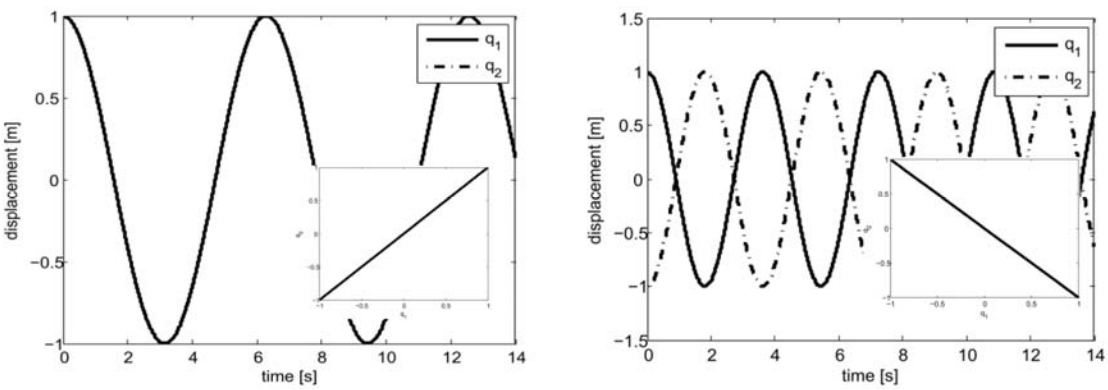

Fig. 2: Time series and motion in configuration space of LNM; (left: in-phase mode T = 6.238 s, right: out-of-phase mode, T = 3.627 s).

Two representations of both LNMs and NNMS are shown in Fig. 2 and 3. Nonlinear normal modes can be represented by time series of coordinates of the system or they can be viewed in configuration space. The LNMs are represented by a straight line in configuration space and NNMs by a general curve. Fig. 2 shows linear normal modes of linearized system (1) while Fig. 3 displays a periodic solution (NNMs) of the nonlinear system (2), which corresponds to free vibration excited by nonzero initial conditions. It can be clearly seen that the nonlinear system can exhibit other periodic solution with different behaviour and time period of motion than those presented in Fig. 2. The periodic solution was found by shooting method combined with Newton-Raphson method. To ensure the convergence of this method, initial conditions have to be very close to existing solution to ensure the sufficient convergence.

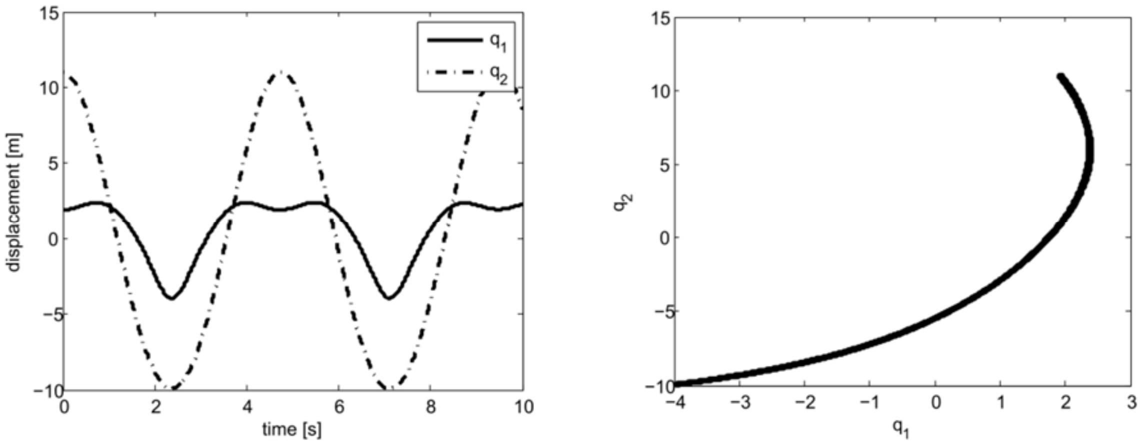

Fig. 3: Time series of NNM motions of system (2) – left, NNM motions in configuration space – right; in-phase NNM for $[q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)] = [1.9, 11, 0, 0]$, *T = 4.75 s.*

3.2. Modal interaction

Another salient feature of nonlinear systems is that NNMs may interact during a general motion of the system. A case of particular interest is when the linear natural frequencies are commensurate or nearly commensurate. An energy exchange between the different modes employed may therefore be observed during the internal resonance. For instance, exciting a high frequency mode may produce a large amplitude response in a low frequency mode. Dynamical interaction of an elastic system and nonlinear absorber exploiting these energy transfers has been studied by Mikhlin & Reshetnikova (2005).

3.3. Mode bifurcations and stability

A third fundamental property of NNMs is that their number may exceed the number of DOFs of the system. Due to mode bifurcation, not all NNMs can be regarded as nonlinear continuation of normal modes of linear system (Kerschen et al., 2009). Internally resonant NNMs are one example, another possible example corresponds to the NNM bifurcation of the system

$$\begin{aligned} \ddot{q}_1 + q_1 + q_1^3 + K(q_1 - q_2)^3 = 0, \\ \ddot{q}_2 + q_2 + q_2^3 + K(q_2 - q_1)^3 = 0 \end{aligned} \tag{2}$$

for variations of the parameters *K*. This system possesses similar NNMs that obey to the relation $q_2(t) = cq_1(t)$. Eliminating $q_2(t)$ from Eqs. (2), one obtains two equations for $q_1(t)$ which must lead to the same solution. Therefore, after some modifications, it follows

$$K(1+c)(c-1)^3 = c(1-c^2), \qquad c \neq 0. \tag{3}$$

Eq. (3) means that system (2) always possesses two modes characterized by $c = \pm 1$ which are direct extension of the LNMs. However, this system can possess two additional similar NNMs which cannot be captured using linearization procedures. At *K=0.25*, these NNMs bifurcate from the out-of-phase mode (see Fig. 4) (Kerschen et al., 2009).

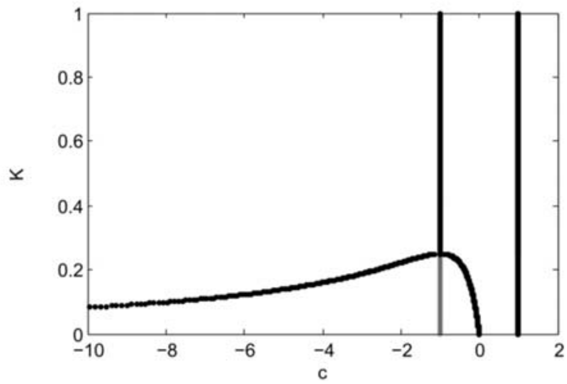

Fig. 4: NNM bifurcation of system (2) (black – stable NNMs, grey – unstable NNM).

4. Computation of NNMs

Numerical calculation is based on the fact that the NNMs are a continuation of LNMS. Therefore, continuation methods are often employed (Peeters et al, 2009). The simplest and most intuitive continuation technique is the sequential continuation method, which uses the shooting method to find the periodic solution for given time period. But this method has some drawbacks. The convergence depends critically on the closeness of the initial guess to the actual solution and it is unable to deal with so called turning points. For better performance, a continuation algorithm uses a more sophisticated prediction than the last computed solution. In addition, corrections of the period have to be considered. The sequential continuation method was used to calculate presented results.

5. Conclusions

This paper deals with nonlinear normal modes (NNMs) as a useful theoretical and mathematical tool for nonlinear system investigation. Nonlinear normal modes represent a continuation of linear normal modes (LNMs) in cases when the linearized models cannot be used. The NNMs introduce a periodic motion and therefore numerical methods for finding periodic solutions can be advantageously employed. This paper serves as a brief summary of basic NNMs definitions and introduces fundamental properties of NNMs regarding conservative nonlinear systems and shows how the NNMs differ from LNMs. The NNMs approach will be extended to damped nonlinear systems and in the future it will be used for dynamical analysis of mechanical systems with clearances in general.

Acknowledgement

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