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A NEW METHOD FOR VECTOR FIELDS DYNAMIC ANALYSIS

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Summary: *A new method for analysis of spatio-temporal data is proposed. This is based on combination of Proper Orthogonal Decomposition and Principal Oscillation Patterns methods. The method provides dynamical characteristics of spatial modes evaluated either on energetic or stability analysis, respectively. Example on TR-PIV results on unsteady separation is shown.*

1. Introduction

Most of interesting phenomena in fluid dynamics are connected with turbulence being substantially 3D and highly dynamical in nature in the same time. To analyze such a phenomenon, appropriate methods of analysis should be applied on experimental data. The classical data analysis methods are not suitable for this purpose, new methods are needed. The Proper Orthogonal Decomposition (POD) and Principal Oscillation Patterns methods (POPs) are among promising ones. In the presented paper a new method involving both above mentioned methods is proposed.

2. The method description

The Proper Orthogonal Decomposition (POD) method has applications in almost any scientific field where extended dynamical systems are involved. Historically, it was introduced in the context of turbulence by Lumley (1967) as an objective definition of what was previously called big eddies and which is now widely known as coherent structures. The POD method is optimal in sense that the series of eigenmodes converges more rapidly (in quadratic mean) than any other representation. Convergence is very fast in the flows in which large coherent structures contain a major fraction of the total kinetic energy. Details could be found e.g. in Tropea et al. (2007) and Uruba (2009).

The calculus of variations reduces this problem of maximization to a Fredholm integral equation of the first kind whose symmetric kernel is the autocorrelation matrix. The method for energetic modes evaluation is based on correlation matrix $\mathbf{R}$ analysis

$$\mathbf{R} = \left\langle \mathbf{u}(t)\mathbf{u}^T(t) \right\rangle, \tag{1}$$

where $\mathbf{u}(t)$ is vector of velocities or velocity fluctuation components in space, brackets denote mean operation in time t. The POD modes are defined as the correlation matrix eigen-

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vectors, the corresponding eigenvalues indicate energetic content.

The next step Bi-Orthogonal Decomposition (BOD) representing itself an extension of the POD. While POD analyses data in spatial domain only, the BOD performs spatiotemporal decomposition.

Aubry et al. (1991) presented the BOD as a deterministic analysis tool for complex spatiotemporal signals. First, a complete two-dimensional decomposition was performed. These decompositions were based on two-point temporal and spatial velocity correlations. A set of orthogonal spatial ("topos") and temporal ("chronos") eigenmodes are to be computed to allow the expansion of the velocity field.

The orthogonal decomposition is optimal in sense of a fast convergence of the expansion with a small number of terms. It should be noticed that the BOD introduces a time-space separation in the velocity field expansion. While the classic orthogonal decomposition POD is based on full two-point space-time correlations and entails space and time-dependent eigenmodes, BOD is closer to analytical and numerical studies where the velocity field is naturally expanded over products of spatial functions and temporal functions.

The decomposition methods allows studying the energy and entropy distribution in both time and space domains. In addition, truncation of the reconstruction formula neglecting high-order low energy content allows us to study dynamics of the given system using discrete methods of dynamical systems analysis in state space. Thus, the low-dimensional model of the real flow optimal from the energetic point of view is obtained. This means, that even a few lower order (that is high-energy) modes could capture principal properties of the entire system dynamics.

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$$\mathbf{u}(\mathbf{x},t) = \sum_k \lambda_k \, \overline{\boldsymbol{\varphi}_k(\mathbf{x})} \, \boldsymbol{\psi}_k(t). \tag{2}$$

The bar denotes complex conjugate, $\boldsymbol{\varphi}_k(\mathbf{x})$ are spatial eigenfunctions topos, $\boldsymbol{\psi}_k(t)$ are temporal eigenfunctions chronos, λ_k are the common eigenvalues. Mathematical details of the BOD method could be found in Aubry et al. (1991).

The Principal Oscillation Patterns (POPs) decomposition is based on modal structures representing temporal or spatial linear evolution dynamics. The method was introduced in climatology to model evolution of meteorological data. Recently, similar method was introduced by DANTEC as Dynamics Mode Decomposition (DMD) method.

Each mode is characterized by a complex frequency involving information on frequency, phase and growth/decay. There are several modifications of the method involving complex or cyclostationary variants.

In general this method is very effective for studying travelling waves, on the other hand, the POPs analysis is unable to resolve standing oscillations.

The basis of the POPs analysis was formulated by Hasselmann (1987) for discrete Markov processes in linearized dynamical systems driven by white noise with application in climatology. The POPs theory is a special case of more general Principal Interaction Patterns (PIPs) method for nonlinear dynamical systems.

In the POPs approach the fluctuating part of Navier-Stokes equation is modeled by Langevin equation for the linear Markov process:

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{A} \cdot \mathbf{u}(t) + \xi, \tag{3}$$

where $\mathbf{u}(t)$ is vector of velocity fluctuations, $\mathbf{A}$ is the deterministic feedback matrix, ξ is noise driving the system which could be interpreted as influence of smaller, unresolved scales. The equation (3) could be rewritten for time lag τ as followed

$$\mathbf{u}(t+\tau) = \mathbf{B} \cdot \mathbf{u}(t) + \xi . \tag{4}$$

Then the POPs are the eigenfunctions of the matrix $\mathbf{B}$. In other words, the POPs are the empirically computed eigenmodes of the system. The Green function $\mathbf{G}$ could be identified

$$\mathbf{G}(\tau) = \exp(\mathbf{B}\tau) = \left\langle \mathbf{u}(t+\tau)\mathbf{u}^T(t) \right\rangle \left\langle \mathbf{u}(t)\mathbf{u}^T(t) \right\rangle^{-1}. \tag{5}$$

The eigenvalues g_k of the Green function $\mathbf{G}$ are related to eigenvalues β_k of the feedback matrix $\mathbf{B}$ as follows:

$$g_k \equiv \exp(\beta_k \tau). \tag{6}$$

and then

$$\beta_k = \frac{\ln(g_k)}{\tau} \tag{7}$$

The real part of eigenvalues β_k characterizes the decay e-fold time τ_{ek} of the POP (more precisely its reciprocal value), this could be interpreted as the decay rate of our ability to predict development of the POPs. This must be negative for stable system. The complex part gives the k-th POPs oscillation frequency:

$$\begin{aligned} f_k &= \frac{\mathrm{Im}(\beta_k)}{2\pi}, \\ \tau_{ek} &= -\frac{1}{\mathrm{Re}(\beta_k)}. \end{aligned} \tag{8}$$

Common eigenvectors form system of POPs. The right eigenvectors $\mathbf{v}_{rk}$ of $\mathbf{G}$ are computed as well as the left eigenvectors $\mathbf{v}_{lk}$ of $\mathbf{G}^T$. Then the deterministic feedback matrix $\mathbf{B}$ could be formed:

$$\mathbf{B} = \sum_k \mathbf{v}_{rk} \beta_k \mathbf{v}_{lk}^T . \tag{9}$$

Note that $\mathbf{B}$ and $\mathbf{G}$ are obtained solely from the time series data $\mathbf{u}(t)$. If the system is well described by a linear Markov process, then our estimate of $\mathbf{B}$ will be independent of the choice of τ. On the other hand, if nonlinear effects are important, then $\mathbf{B}$ will vary significantly with τ.

The matrix $\mathbf{B}$ is real non-symmetrical, so eigenvalues β and corresponding eigenvectors $\mathbf{v}$ are complex and the complex conjugate β^*, $\mathbf{v}^*$ satisfy the eigenequation as well. In most cases, all eigenvalues are different and the eigenvectors form a linear basis. So each state $\mathbf{u}(t)$ may be uniquely expressed in terms of the eigenvectors

$$\mathbf{u}(t) = \sum_k z_k \mathbf{v}_k \,. \tag{10}$$

The coefficients of the pairs of conjugate complex eigenvectors are conjugate complex, too. Inserting (10) into (4) we find that the coupled system (4) becomes uncoupled, yielding n single equations, where n is the dimension of the process $\mathbf{u}(t)$

$$z(t+1)\cdot\mathbf{v} = \beta\cdot z(t)\cdot\mathbf{v}\,, \tag{11}$$

so that if $z(0)=1$ than

$$z(t)\cdot\mathbf{v} = \lambda^t\cdot\mathbf{v}\,. \tag{12}$$

Then, contribution $\mathbf{V}(t)$ of the complex conjugate pair $\mathbf{v}$, $\mathbf{v}^*$ to the process $\mathbf{u}(t)$ forms the POP signal and it is given by

$$\mathbf{V}(t) = z(t)\cdot\mathbf{v} + \left[z(t)\cdot\mathbf{v}\right]^*. \tag{13}$$

The complex quantities could be written as follows

$$\begin{aligned} \mathbf{v} &= {}^r\mathbf{v} + {}^i\mathbf{v}\cdot i, \\ 2z(t) &= {}^r z(t) + {}^i z(t)\cdot i. \end{aligned} \tag{14}$$

The contribution is

$$\mathbf{V}(t) = {}^r z(t)\cdot{}^r\mathbf{v} + {}^i z(t)\cdot{}^i\mathbf{v} = \rho^t\cdot\left[\cos(\eta t)\cdot{}^r\mathbf{v} - \sin(\eta t)\cdot{}^i\mathbf{v}\right], \tag{15}$$

where $\beta = \rho\cdot\exp(-i\eta)$ for $z(0)=1$. The geometric and physical meaning of (6) is that between the spatial patterns ${}^r\mathbf{v}$ and ${}^i\mathbf{v}$ the trajectory $\mathbf{V}(t)$ performs a spiral (Fig. 1) with period $T = 2\pi/\eta$ and e-folding time $\tau = -1/\ln(\rho)$, in the consecutive order

$$\ldots\to{}^i\mathbf{v}\to{}^r\mathbf{v}\to-{}^i\mathbf{v}\to-{}^r\mathbf{v}\to{}^i\mathbf{v}\to{}^r\mathbf{v}\to\ldots \tag{16}$$

In Fig. 1 typical evolution the POPs signal is shown for ${}^r z(0)=0$ and ${}^i z(0)=1$. In this demonstration (from [17]) the period is $T\approx 9$ and the e-folding time is $\tau_e \approx 2{,}8$.

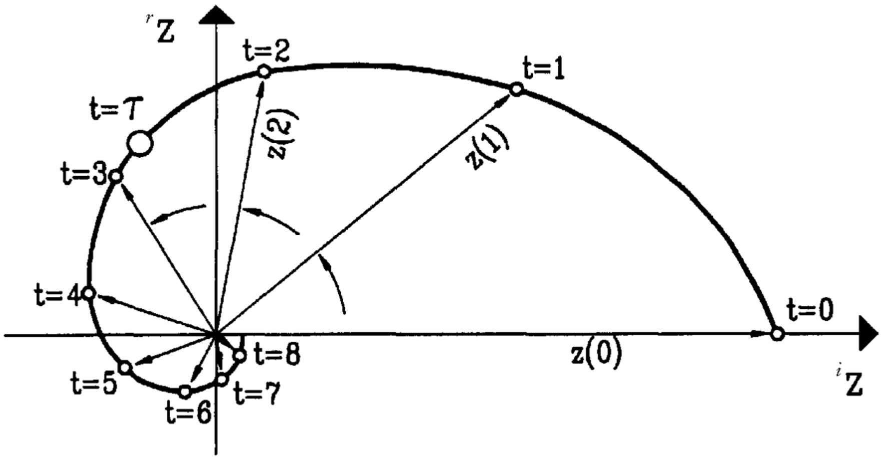


Fig.1 – Time evolution of the POP signal

The e-folding time τ_e has to be considered with some caution. It represents formally the average time for an amplitude of strength one to reduce to $1/e$. But in the POPs context this time is a statistic of the entire time interval, i.e., it is derived not only from the episodes when the signal is active but also from those times when the signal is weak or even absent. As such, the mode will be dampened less quickly as indicated by the e-folding time when the mode is active. The other limitation refers to the presence or absence of high-frequency variations. If these are filtered out, the e-folding time is lengthened.

The pattern coefficients $z(t)$ are given as the dot product of $\mathbf{u}$ and adjoin patterns $\mathbf{v}_j^A$, which are normalized eigenvectors of the matrix $\mathbf{B}^T$:

$$z_j = \left(\mathbf{v}_j^A\right)^T \mathbf{u} = \sum_k z_k \left(\mathbf{v}_j^A\right)^T \mathbf{v}_k \; . \tag{17}$$

The numerical implementation of the POPs method is generally very hardware demanding. On the other hand, there is no need to analyze all modes, only those which are the less stable are of practical importance. The idea is to compute only "important" modes – e.g. those which are not very stable and containing significant portion of energy in the same time.

The proposed method uses both POD and POPs approach. To reduce the number of spatial degrees of freedom in some applications, the data are subjected to a truncated POD expansion, and the POPs analysis is applied to the vector of the first POD coefficients. A positive by-product of this procedure is that noisy components can be excluded from the analysis. Then, the covariance matrix has a diagonal form.

If there is a priori information that the expected signal is located in a certain frequency band, it is often advisable to time-filter the data prior to the POPs analysis. A somewhat milder form of focusing on selected time scales is to derive the POD modes from time-filtered data and then to project the unfiltered data on these modes.

The proposed method have been tested on the case of a boundary layer separation in adverse pressure gradient – see Uruba (2008), Uruba, Sedlák (2009), Uruba (2009a) and Uruba (2009d) and dynamics of a wake behind Ahmed body – see Uruba (2009b).

2. Application of the method

The above described method is to be applied to the case of a boundary layer separation in adverse pressure gradient. The experiments were carried out on model of a boundary layer in adverse pressure gradient in the IT AS CR.

The blow-down aerodynamic rig has been used for the experiment. The test section for generation of adverse pressure gradient in channel was designed and manufactured. Experiments are described in Uruba (2008).

In Fig.2 the schema of experimental setup is shown. The coordinate system was defined with the x axis in the input flow direction on the wall and y axis is perpendicular to the wall. Origin of the coordinates is in the beginning of the diffuser, the cross-section here is 100 x 100 mm^2. Downstream of this section, the upper wall is inclined with angle $\alpha = 16°$, while the bottom plane wall is used to study the boundary layer separation. To prevent separation from the upper wall, this is permeable and sucked out. The mean flow velocity outside boundary layer at $x = 0$ was 12.4 m/s, the boundary layer was of turbulent nature, about 5 mm thick. The suction velocity along the upper wall could be estimated to 5 m/s.

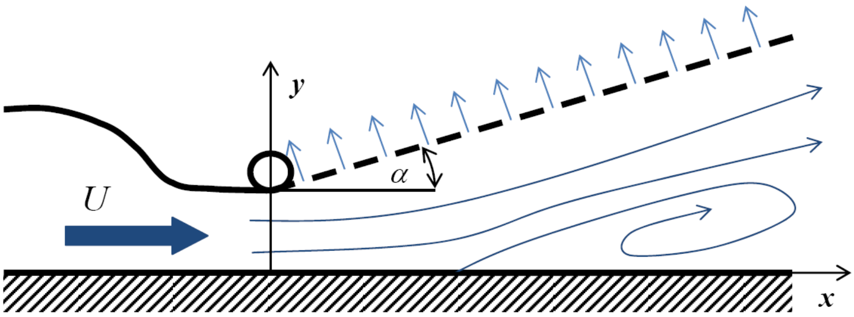


Fig.2 – Experimental setup

The time-resolved PIV method was used to resolve the instantaneous velocity fields. The measuring system DANTEC consists of laser with cylindrical optics and CMOS camera. Laser New Wave Pegasus Nd:YLF, double head, wavelength 527 nm, maximal frequency 10 kHz, a shot energy is 10 mJ for 1 kHz (corresponding power 2 x 10 W). The used camera NanoSense MkIII with reduced, resolution 1280 x 350 pixels was used on 1500 double-snaps per second. The camera internal memory 4 GB represents 5000 double-snaps. The software Dynamics Studio 2.2 was used for velocity-fields evaluation. All measurements were carried out in the *xy* plane of symmetry. Detailed description of the experimental setup is given in Uruba (2008).

For dynamical analysis only fluctuating parts of all velocity components were considered (unlike in Uruba, 2008, where entire velocity vectors are considered). The mean flow is shown in Fig.3 representing mean vector field and longitudinal velocity component distribution.

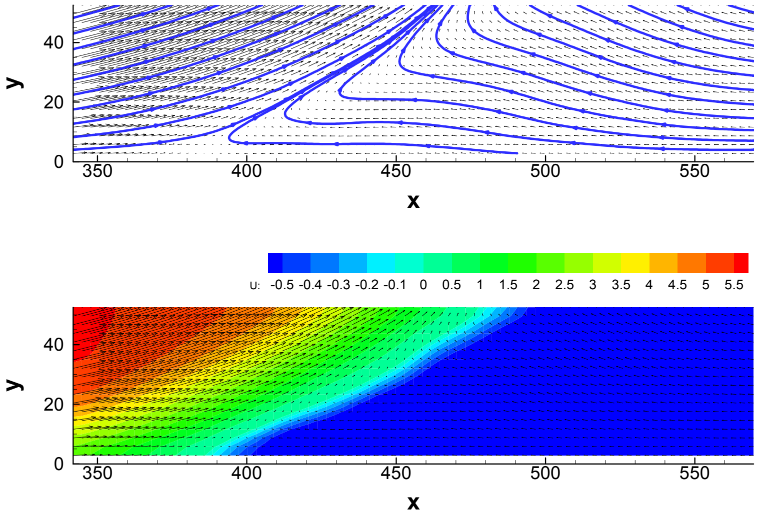


Fig.3 – Mean velocity field

From the Fig.3 using the classical definition of a boundary layer separation point its position could be estimated to be 380 mm approximately.

The BOD modes are arranged according to the energy fraction contained in the individual mode in descending order. In Fig.4 cumulative relative energies for individual modes are shown. The 10 lower order modes represent about 65 % of the total fluctuation energy, while 100 modes represent even more then 94 %.

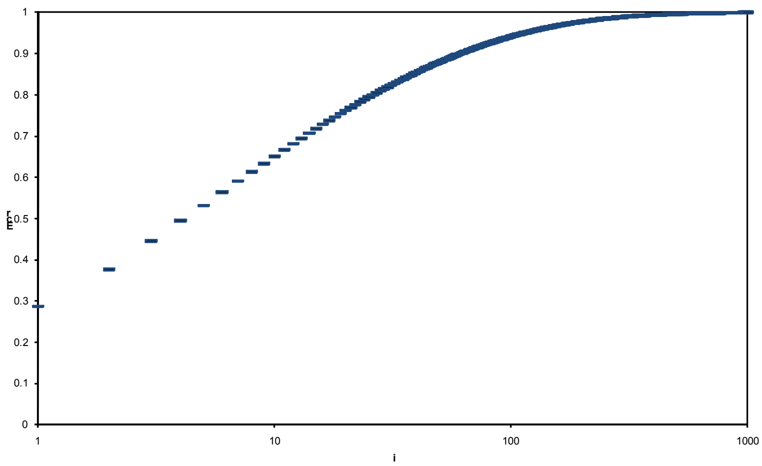


Fig.4 – Cumulative energy of POD modes

The new dynamical systems has been defined using a limited number of POD modes n_{POD} = 10, 15, 20, 30, 50, 100 and 500 most energetic modes corresponding to cumulative energies 65, 72, 76, 82, 88, 94 and 99,6 % respectively. Then the POPs analysis has been applied to those systems accordingly.

The eigenvalues β_k of the feedback matrix $\mathbf{B}$ are evaluated for the dynamical models defined above. The spectrum for the most complicated model $n_{\mathrm{POD}} = 500$ containing all 500 modes is shown in Fig.5. Please note that the POPs analysis gives complex conjugate eigenvalues (or real). In the bottom a few eigenvalues with biggest (but negative) real part are presented, which are the most important from practical point of view.

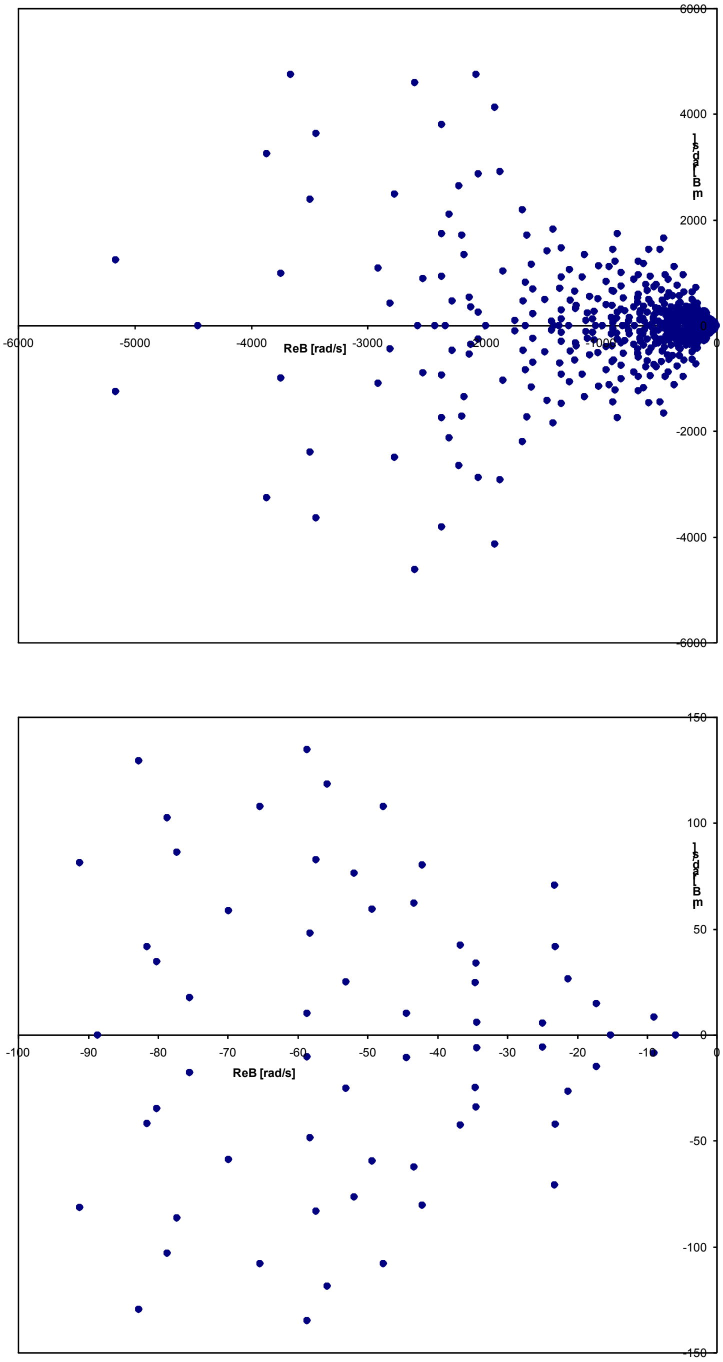


Fig.5 – POPs spectrum for 500 POD modes

Then, the mode frequencies and e-fold times have been evaluated for all models in question using formulas (8). The time lag was considered to be one acquisition period, that is 0,00067 s.

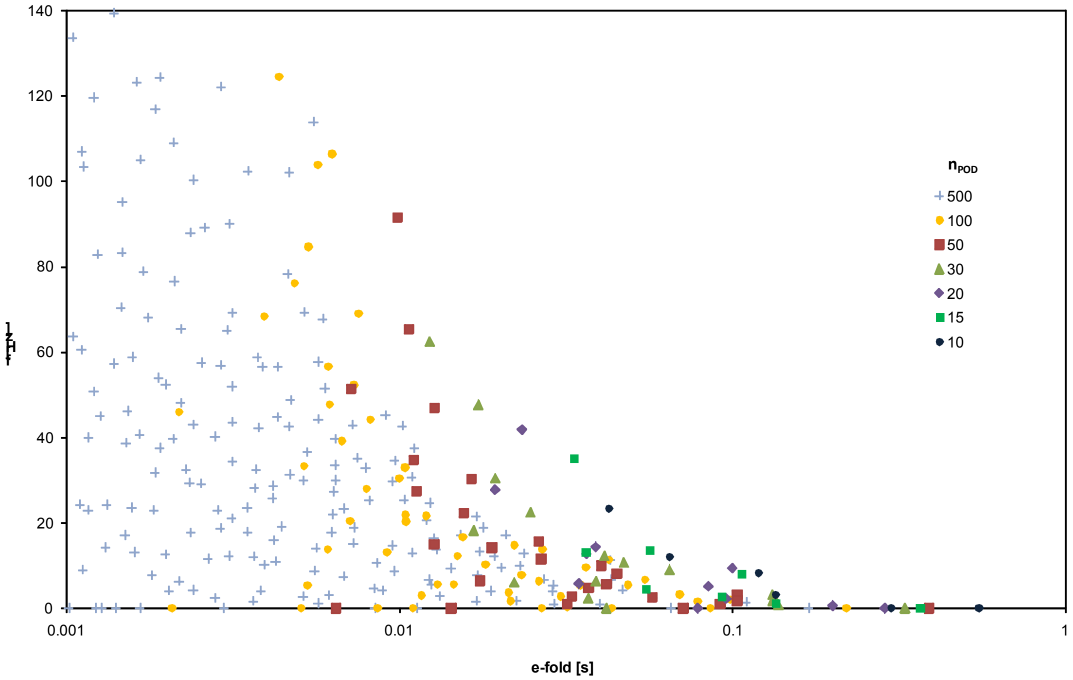


Fig.6 – POPs modes frequencies and e-fold times

All evaluated POPs modes was stable, however some are oscillating and others non-oscillating (characterized by zero imaginary part of the eigenvalue). The evaluated characteristics of modes with the biggest e-fold time are similar for all models, difference could be recognized in presence of additional modes characterized by lower e-fold time values in the case of higher order models. The mode with highest e-fold time value is for all models characterized by non-oscillating type.

Influence of time lag τ was studied as well. The basic value of time lag was chosen as the acquisition period, test for double period has been performed. In Fig.7 there is comparison of the case model $n_{POD} = 100$ and time lag τ is 1 and 2 acquisition period dt respectively. The difference between calculated spectra is not very important, however for the shorter time lag the range is more extended.

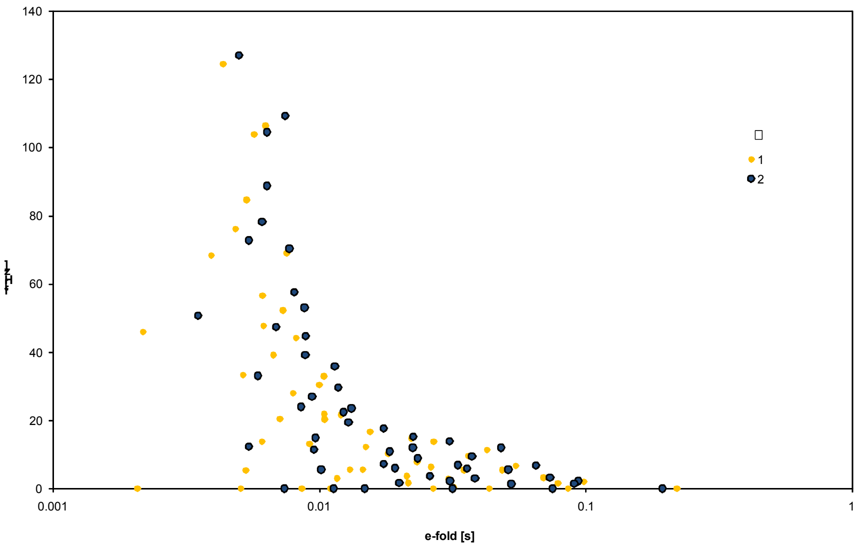


Fig.7 – POPs modes frequencies and e-fold times for two time lags and n_{POD} = 100

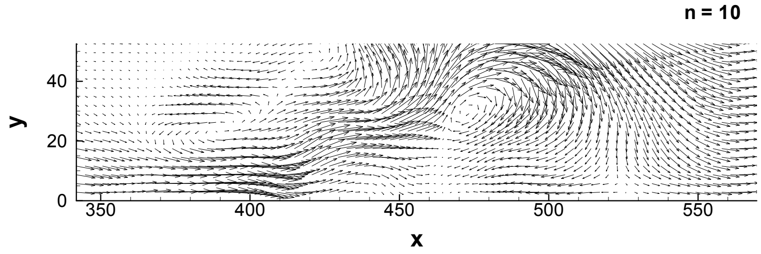
We analyzed in detail the less stable mode which is in all cases of non-oscillating type. The mentioned modes are of non-oscillating type with zero imaginary eigenvalue part. The e-fold times evaluated for those modes for various n_{POD} is given in Tab.1.

n_{POD}	e-fold time [ms]
10	550
15	367
20	287
30	330
50	389
100/1	220
100/2	197
500	169

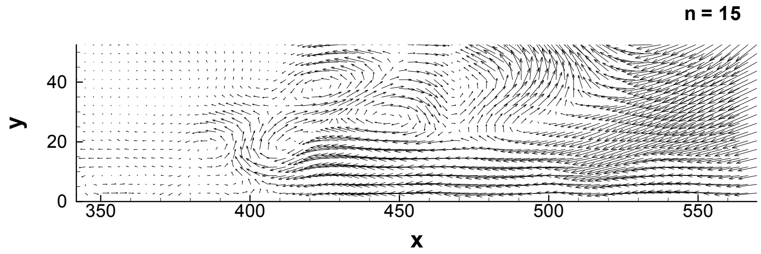
Tab.1 – E-fold time of the less stable mode

The corresponding e-fold time ranges from 169 ms for n_{POD} = 500 to 550 ms for n_{POD} = 10. The POPs modes topology differs as well. Comparison of POPs modes of the less stable mode for various n_{POD} and time lag τ for is shown in Fig.8.

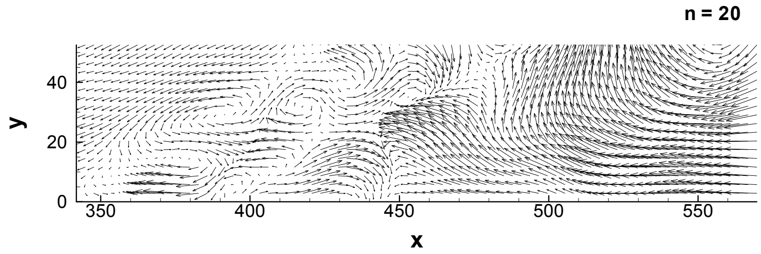
$n_{\mathrm{POD}} = 10$
$\tau = 1dt$



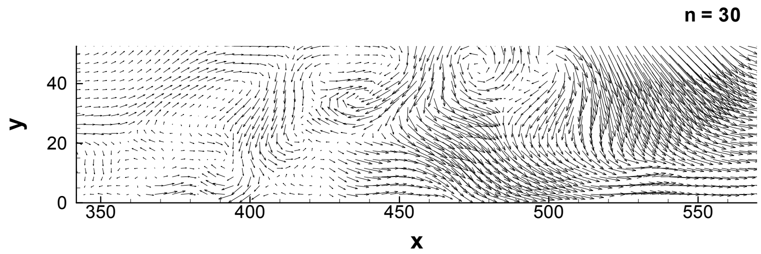
$n_{\mathrm{POD}} = 15$
$\tau = 1dt$



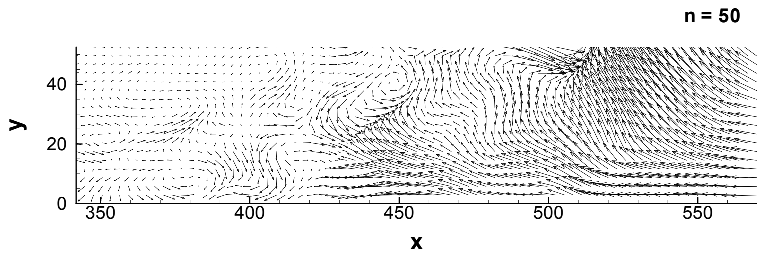
$n_{\mathrm{POD}} = 20$
$\tau = 1dt$

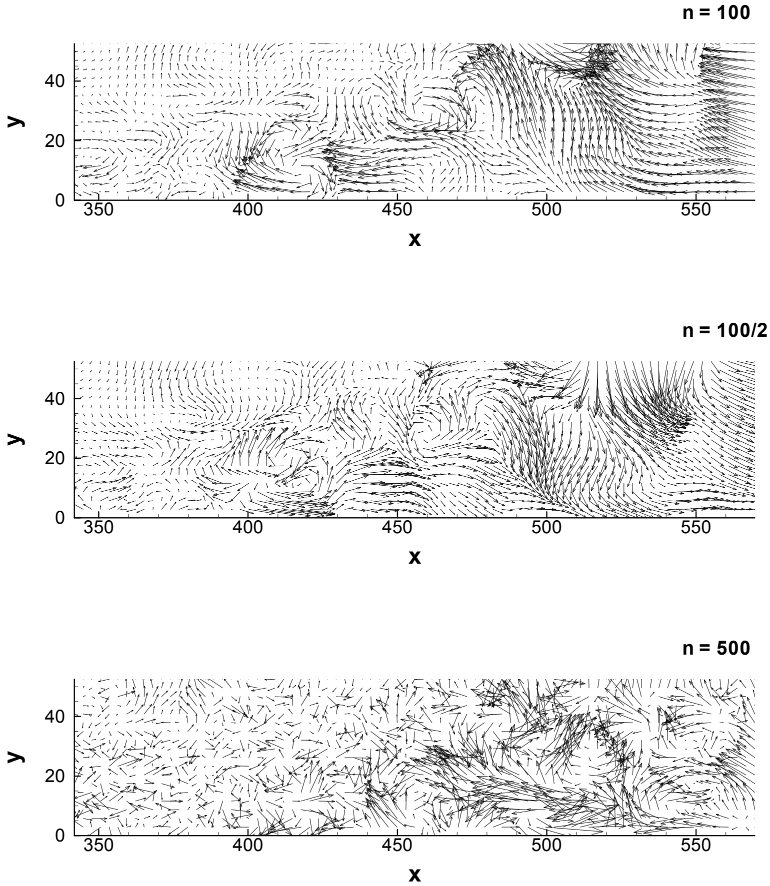


$n_{\mathrm{POD}} = 30$
$\tau = 1dt$



$n_{\mathrm{POD}} = 50$
$\tau = 1dt$





$n_{\mathrm{POD}} = 100$
$\tau = 1dt$

$n_{\mathrm{POD}} = 100$
$\tau = 2dt$

$n_{\mathrm{POD}} = 500$
$\tau = 1dt$

Fig.8 – POPs modes of the less stable mode for various n_{POD} and time lag τ

The POPs modes are characterized by system vortices along the shear layer. The position of the shear layer is very distinct in the mean velocity distribution Fig.3 being defined as a line with zero mean longitudinal velocity component. Difference between the models is in complexity of the shear layer region and significance of the chaotic component. The first approach $n_{\mathrm{POD}} = 10$ provides very simplified dynamics, while full dynamical content characterized by number of modeled vortices in shear layer is reached for $n_{\mathrm{POD}} = 30$, for higher n_{POD} chaotic behavior is more important. The case $n_{\mathrm{POD}} = 500$ is highly chaotic. Influence of time lag is for a given case unimportant.

3. Conclusion

The new method of dynamic analysis of vector fields has been proposed. The method is based on POD, BOD and POPs methods. The reduction of the number of degrees of freedom is performed on energetic principle using POD or BOD method. Then the stability analysis of reduced dynamical system is performed.

The POPs modes differ according to number of POD modes involved. However, some common feature could be recognized. So, even the simplest dynamical model with 10 degrees of freedom provides valuable information about dynamical behavior of the complete system.

4. Acknowledgement

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