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EXPERIMENTAL AND THEORETICAL STABILITY ANALYSIS OF DAMPED AUTO-PARAMETRIC PENDULUM

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Summary: *Engineering structures subjected to dynamic loading are often equipped with special devices proposed to suppress adverse motion. However, these installations are often designed with certain stereotyped approach, considering only simplified linear single-degree-of-freedom (SDOF) model which in practice is often not sufficient. Typical examples, where the practical results differ from theoretical design are dynamic absorbers on towers. These dampers are often designed as a pendulum supposed to be effective in one direction only, ceasing however its functionality due to neglect of some features. The article describes an experimental and theoretical treatment of the auto-parametric pendulum damper modelled as a double-degree-of-freedom (DDOF) system. It is strongly non-linear, with the harmonic excitation being applied in the suspension point and interpreted for instance as an excitation by wind. The stability of the motion in a vertical plane is analysed in the meaning of semi-trivial vibration. This effect of stability loss and possible stability regain is very important from practical point of view, because the motion of an damper beyond the stability limits may act negatively and thus endanger the structure itself. Special experimental frame was developed. It contains a pendulum supported by the cardan joint and excited by a shaker. There are two magnetic units attached to the frame and to the supporting axes of rotation. The are able to reproduce linear viscous damping for both degrees of freedom. The stability of the system is analysed experimentally and compared with theoretical results.*

1. Introduction

Many structures encountered in the civil and mechanical engineering are equipped by devices reducing dynamic response component due to external excitations of deterministic or random character. As examples can be mentioned vertical slender structures like towers, masts, chimneys, etc., exposed to strong dynamic wind effects, or power piping systems, massive foundations of rotating machines, special transport means, and other general purpose systems subjected to strong excitation of technological origin. A lot of monographs, for instance Hartog (1956), and papers dealing with various aspects of this topics have been published until now, e.g. Naprstek (1998); Pospisil (2006); Warminski (2005) and many others. In Berlioz (2000) for example, the experimental and numerical investigations are carried out on an auto-parametric system consisting of a composite pendulum attached to a harmonically base excited mass-spring subsystem.

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The pendulum dampers and similar low cost passive devices are still very popular for their reliability and simple maintenance, see e.g. Naprstek (2002). However dynamic behaviour of a pendulum is significantly more complex than it is supposed by widely used simple linear SDOF model working in the vertical planes (xz) or (yz), which are considered independent, see Fig. 1. The conventional linear model is satisfactory only if the amplitude of kinematic excitation $a(t)$ at the suspension point is very small and if its frequency remains outside a resonance frequency domain. This is however a case of lower efficiency of the damper.

The spherical pendulum should be considered as a non-linear system of the auto-parametric type. Its movement is described in two coordinates θ, φ on a spherical surface respecting the non-linear interaction of both components or in three Cartesian coordinates x, y, z with one geometric constraint. It means that the system response is described in the simplest case by two simultaneous differential equations of the second order which are independent on the linear level. Their interaction follows from non-linear terms. Depending on its parameters various types of a stability loss can occur and, consequently, a critical amplitude and frequency of the excitation can be easily encountered. Therefore they can admit the semi-trivial solution when one component is non-trivial while the second one remains trivial. Under certain conditions the semi-trivial solution can lose its stability and various specific types of the response can occur.

Stable and unstable, stationary and non-stationary post-critical states in the resonance domain can occur. At the lower limit of this domain the system is forced to abandon the planar response in the vertical plane (xz) and to follow complicated space trajectories of different types. This figure stabilizes for increasing frequency in a nearly elliptic "horizontal" trajectory. Above the upper limit of the resonance domain an existence of a stable deterministic solution in the vertical plane resumes. The existence and stability level of individual solutions or response types are dependent on pendulum geometry and excitation structure. Wide-band excitation especially of a random character can lead to response abandoning the expected plane movement in a real practice. Then the pendulum loses its purpose and can influence the structural system negatively.

Auto-parametric systems have been intensively studied for three last decades. Theoretical outline dealing with these systems have been presented probably for the first in Haxton (1972). During further period many papers contributing to analytical, numerical as well as experimental aspects of auto-parametric systems have been published mostly by Tondl and co-authors, e.g. Nabergoj (1994)-Tondl (1997) and many others. Several monographs, e.g. Tondl-etal (2000), presenting a comprehensive overview of results and methods appeared until now. Motivation for these studies comes from various areas of mechanical and civil engineering. The work in this article treats experimentally the analytical approach to the subject which is described in Naprstek (2009).

2. THEORETICAL BACKGROUND

The spherical pendulum will be considered as strongly non-linear dynamics system with kinematic external excitation in the suspension point.

The mathematical model of the pendulum according to Fig. 1 follows from the mechanical energy balance. However, in order to formulate a solution of the system as a semi-trivial solution in the plane (xz) with a small perturbation orthogonal to this plane, governing system should be written in components $\xi(t), \zeta(t)$ corresponding with Cartesian coordinates x, y. The

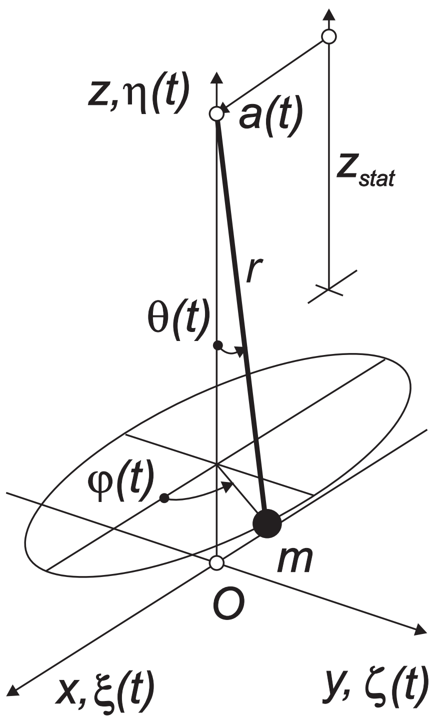


Figure 1: Schematic of the system.

case when the rotation $\varphi(t)$ vanishes makes together with non-zero $\theta(t)$ the semi-trivial solution. Any small perturbation cannot be formulated in $\theta(t), \varphi(t)$ variables, because $\varphi(t)$ changes very quickly when $\theta(t)$ is in a neighbourhood of zero. In principle the non-perturbed (xz) planar trajectory the component $\varphi(t)$ changes suddenly in π when $\theta(t)$ is going through zero. Consequently $\varphi(t)$ cannot be defined as a function with arbitrarily small norm and subsequently to be limited to zero.

Let's write kinetic and potential energies T, V in basic x, y, z system $(\xi(t) = \xi, \zeta(t) = \zeta, \eta(t) = \eta)$ with the kinematic constraint, relevant to the physical model illustrated in Fig. 1:

$$\left.\begin{array}{rcll} T &=& \frac{m}{2}(\dot{\xi}^2 + \dot{\zeta}^2 + \dot{\eta}^2 + 2\dot{u}\dot{\xi} + \dot{a}^2) & (a) \\ V &=& mg\eta & (b) \\ & & \xi^2 + \zeta^2 + (r-\eta)^2 = r^2 & (c) \end{array}\right\} \qquad (1)$$

where vertical coordinate η starts in the lower pole of the sphere and m, r are mass and suspension length of the pendulum, respectively and $a = a(t)$ is amplitude of kinematic excitation in the suspension point.

After application of Hamilton's principle and some simplification, an approximate Lagrangian system in x, y coordinates for components ξ, ζ on the level $O(\varepsilon^6)$; $\varepsilon^2 = (\xi^2 + \zeta^2)/r^2$ can be obtained. The damping should also be included in order to enable analysis of the true stationary response. The so-called Rayleigh function is used in a form $F = m\omega_b r^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)$ to derive the damping terms, ω_b is a coefficient of viscous damping being taken equal in both coordinates. Natural frequency of the pendulum is given by $\omega_0^2 = g/r$. The final form of the differential system reads (see Naprstek (2009) for full derivation) :

$$
\left.\begin{array}{lcll}
\ddot{\xi} + \frac{1}{2r^2}\xi\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\omega_b\dot{\xi} + \omega_0^2\left(\xi + \frac{1}{2r^2}\xi(\xi^2+\zeta^2)\right) & = & -\ddot{a} & (a) \\
\ddot{\zeta} + \frac{1}{2r^2}\zeta\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\omega_b\dot{\zeta} + \omega_0^2\left(\zeta + \frac{1}{2r^2}\zeta(\xi^2+\zeta^2)\right) & = & 0 & (b)
\end{array}\right\} \quad (2)
$$

The above equations are mutually independent on the level of the linear approximation. Their interaction is given by non-linear terms only. Each of the response components ξ, ζ can be separately considered as arbitrarily small in the norm and independently and continuously limited to zero, while the other one remains non-trivial. Therefore the system is auto-parametric and respective procedures can be applied.

A basic type of motion takes course in the vertical (xz) plane if the time history starts in homogeneous initial conditions. With increasing amplitude of the excitation $a(t)$ the auto-parametric stability loss can occur and the post-critical state of the auto-parametric resonance arises. To analyse the critical state the semi-trivial solution stability should be assessed. It means a neighbourhood of the semi-trivial solution is to be investigated observing whether a small perturbation has a tendency to decay or increase. For details of dynamic stability investigation in deterministic domain, see for instance Guckenheimer (1983) and many others.

To investigate the semi-trivial solution, let us assume the excitation to be harmonic:

$$
a(t) = a_0 \sin\omega t \quad (3)
$$

and the solution to have form of:

$$
\xi_0 = a_c \cos\omega t + a_s \sin\omega t \quad (4)
$$

Obviously, as we are seeking the semi-trivial solution, we assume that $\zeta = 0$. After substitution of this assumption and Eq. (3) into Eqs (2), the Eq. (2b) is fulfilled identically while Eq. (2a) gives:

$$
\ddot{\xi}(1+\frac{\xi^2}{r^2}) + \frac{1}{r^2}\xi\dot{\xi}^2 + 2\omega_b\dot{\xi} + \omega_0^2\xi(1+\frac{1}{2r^2}\xi^2) - a_0\omega^2\sin\omega t = 0 \quad (5)
$$

Coefficients a_c, a_s in (4) should be considered as functions of time: $a_c = a_c(t)$, $a_s = a_s(t)$. However, if a stationary solution exists for a given excitation frequency ω, then a_c, a_s should converge to constants for increasing $t \to \infty$. On the other hand, in certain frequency intervals the solution can have strongly non-stationary character. Amplitudes can follow either periodic or non-periodic curve. For this reason coefficients a_c, a_s can be considered as constants only under special conditions when stable stationary response can be expected, see also Arnold (1978).

Putting Eq. (4) in Eq. (5) and applying the operation of the harmonic balance, the following algebraic system can be obtained:

$$
\left.\begin{array}{lcll}
a_c\left((\omega_0^2-\omega^2) + \frac{1}{2r^2}\left(\frac{3}{4}\omega_0^2-\omega^2\right)(a_c^2+a_s^2)\right) + 2\omega\omega_b\cdot a_s & = & 0 & (a) \\
a_s\left((\omega_0^2-\omega^2) + \frac{1}{2r^2}\left(\frac{3}{4}\omega_0^2-\omega^2\right)(a_c^2+a_s^2)\right) - 2\omega\omega_b\cdot a_c & = & a_0\cdot\omega^2 & (b)
\end{array}\right\} \quad (6)
$$

Both equations should be raised to the second power and summed together. Finally the equation for the amplitude of the response $R_0^2 = a_c^2 + a_s^2$ arises:

$$R_0^2 \left[4\omega^2\omega_b^2 + 4\left((\omega_0^2 - \omega^2) + \frac{1}{2r^2}\left(\frac{3}{4}\omega_0^2 - \omega^2\right) R_0^2 \right)^2 \right] - \omega^4 a_0^2 = 0 \tag{7}$$

The Eq. (7) is a cubic equation for R_0^2. Increasing a_0 or ω_b as an excitation amplitude or damping parameter a set of resonance curves can be obtained as functions of the excitation frequency ω. As it is well known these functions can lose their unique character in some intervals of ω. It is related with an existence of one or three real roots of Eq. (7) for particular values of the parameters a_0, ω_b. Fig. 2a demonstrates the resonance curve (thick solid line) for the in-plane motion of the pendulum. The thin solid line shows the resonance curve of ξ variable obtained from the numerical solution of the Eq. (2). Parameters of the pendulum are as follows: $r = 0.368\,m, g = 9.81\,ms^{-2}, \omega_0 = 5.16\,s^{-1}, a_0 = 0.005\,m$. The validity of the figure should be understood as limited in the grey part, where the assumptions of the semi-trivial solution are not fulfilled. However, outside of the unstable region both resonance curves coincide. Nevertheless "softening" character is visible as well as domains of ω where these curves lose an unique character.

To assess the stability of the semi-trivial solution, let us endow the semi-trivial solution (4) with small (in a meaning of a norm) perturbations u, v in both coordinates:

$$\left.\begin{array}{lllllll} \xi & = & \xi_0 & + & u & \qquad (u = u(t)) & \qquad (a) \\ \zeta & = & 0 & + & v & \qquad (v = v(t)) & \qquad (b) \end{array}\right\} \tag{8}$$

After inserting expressions (8) and (4) into Eqs (2), one realises that ξ_0 is the solution of the simple Eq. (2a) and that only the first powers of u, v and their derivatives should be kept. In the neighbourhood of stability limits both perturbations can be written in a form:

$$\left.\begin{array}{lll} u(t) & = & u_c \cos\omega t + u_s \sin\omega t \qquad (a) \\ v(t) & = & v_c \cos\omega t + v_s \sin\omega t \qquad (b) \end{array}\right\} \tag{9}$$

Approximations (9) are to be substituted into the Eqs (2) and operation of the harmonic balance is applied once again. After some tedious algebra, one obtains two homogeneous independent linear algebraic systems for u_c, u_s and v_c, v_s, both of them are free of the excitation amplitude.

To receive a non-trivial solution for u_c, u_s or v_c, v_s, the determinant of the either system must equal zero. This condition leads to two independent equations:

$$\left.\begin{array}{l} \frac{1}{2r^2}R_0^2\left(\frac{3}{4}\omega_0^2 - \omega^2\right)\left(4(\omega_0^2 - \omega^2) + \frac{3}{2r^2}R_0^2\left(\frac{3}{4}\omega_0^2 - \omega^2\right)\right) + \\ \qquad + (\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_b^2 = 0 \qquad (a) \\ \frac{1}{2r^2}R_0^2\left(\omega_0^2(\omega_0^2 - \omega^2) + \frac{1}{2r^2}R_0^2\left(\frac{1}{4}\omega_0^2 + \omega^2\right)\left(\frac{3}{4}\omega_0^2 - \omega^2\right)\right) + \\ \qquad + (\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_b^2 = 0 \qquad (b) \end{array}\right\} \tag{10}$$

The Eqs. (10) act as limits separating the plane (R_0^2, ω) into the stable and unstable domains, see Fig. 2. This figure summarises resonance curves and stability limits of both types as they are

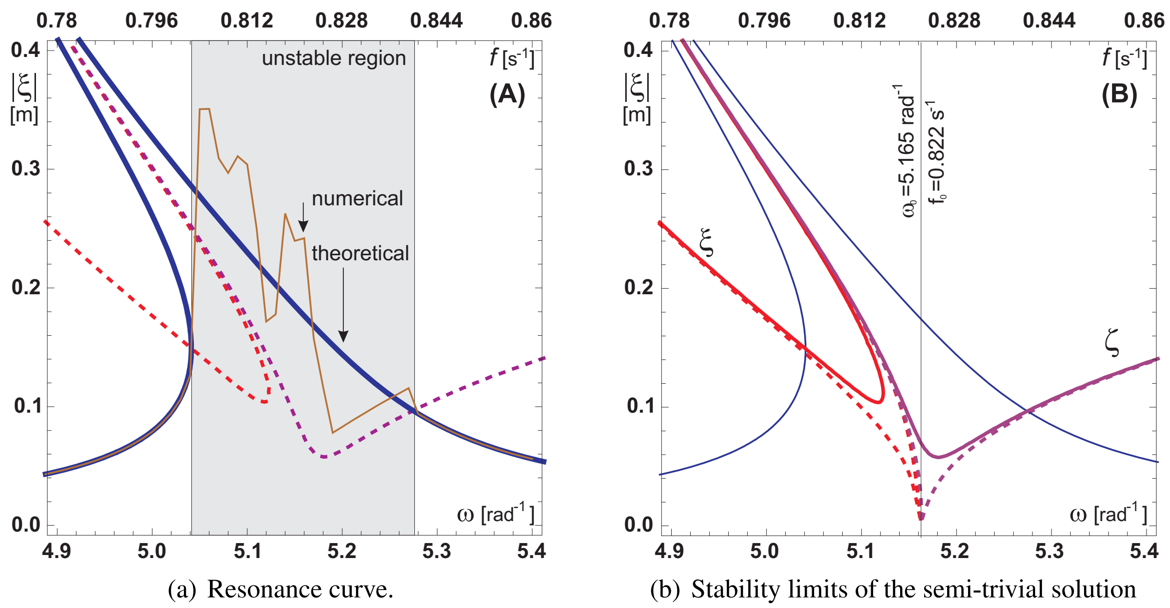


Figure 2: (a) Thick solid line is computed according to Eq. (7), thin solid line is obtained from the numerical solution of the Eq. (2). Parameters of the pendulum: $r = 0.368m, g = 9.81ms^{-2}, \omega_0 = 5.16s^{-1}, \omega_b = 0.024, a_0 = 0.005m$. (b) Stability limits of the semi-trivial solution for various values of the damping factor: (ζ) out of (xy) plane, (ξ) in (xy) plane. The dashed curves correspond to non-damped case ($\omega_b = 0$).

provided by Eqs (7) and (10). Solid line in the part (b) labelled as (ζ) concerns the stability limits of coordinate ζ when the primary movement proceeds in (xy) plane. The curve may be called ζ stability limit. To cross over the curve from the left or bellow means that the coordinate ζ loses its zero stable value and the system response has a tendency to take the form of a spatial curve. In order to abandon the domain of instability and to resume the stable state it is either needed to decrease the excitation amplitude bellow this limit or to increase/decrease the frequency.

Solid line in Fig. 2b labelled as (ξ) represents limit where the response for particular a_0, ω_b is not unique and the resonance curve has two stable and one unstable parts as it is usual when studying non-linear systems.

In general, mutual position of the resonance curve and both stability limits depends on the structural and excitation parameters $(r, \omega_0, \omega_b, a_0)$. For small excitation amplitude a_0 or large enough damping ω_b, the curves do not intersect and the instability interval can disappear completely. On the other hand, for large excitation amplitude and/or small damping can the instability interval occupy a wide frequency domain. It implies the design recommendation, that the damping should not drop bellow a certain limit to avoid the instability which implies the decrease of the pendulum efficiency especially for broad band excitations. In any case the ξ instability is more important than the ζ stability and therefore it should be preferentially avoided, because the ξ instability leads to chaotic response of the pendulum and to the drop of its efficiency.

Let us recall an important fact, that the whole analysis above is based on the supposition of the stationary system response. The interval of frequencies ω with an unstable response should be subjected to detailed analysis. The post-critical behaviour is needed to be thoroughly investigated. In order to describe the real character of the system response in this interval (if it exists), it is necessary to realise that a periodic or non-periodic non-stationary response can arise. As regards the resonance curve, it gives no information about the response in this interval. It can be used only as an indicator of a certain initial state before the post-critical state starts to stabilise in any of the post-critical regimes.

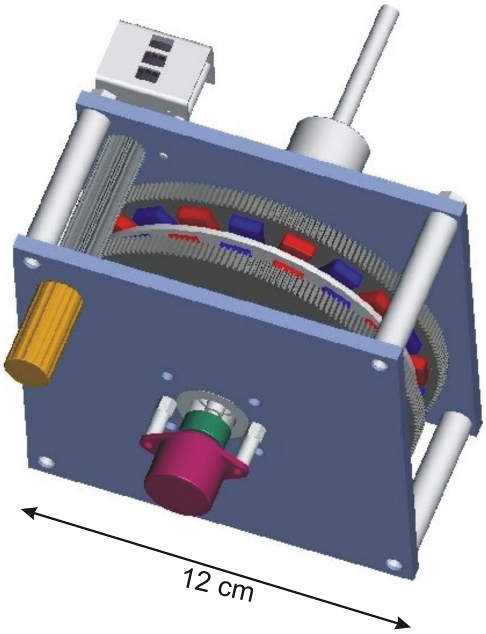
3. Experiments

Any theory has to be verified by repeated experiments in order to see its validity. This applies especially to the non-linear mechanics, where the problems are very complicated due to the number of entering parameters. In this section, an experimental gadget designed for the analysis of the semi-trivial stability is described. The results are compared with semi-analytical approach.

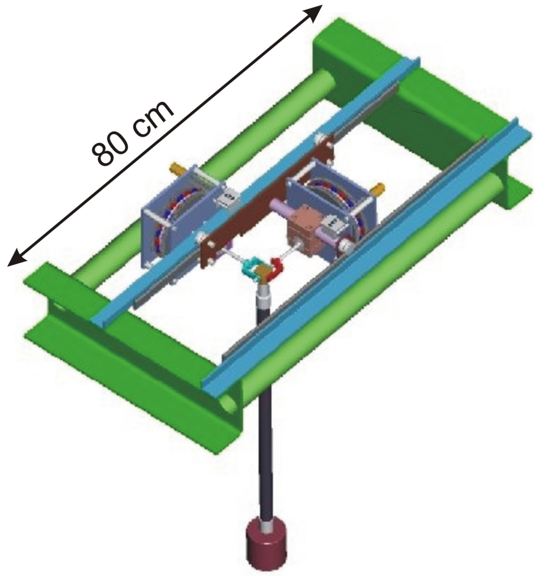
3.1. Laboratory set-up

The stability problems in general are very sensitive to boundary and initial condition. Therefore, any simulation machine and its mechanical parts needs to be well prepared and manufactured to avoid creating parasitical influences which are very difficult to eliminate. This applies not only to a complicated kinematic mechanism but also to the relatively simple spherical pendulum analysing the DDOF system. The authors present here a experimental pendulum, designed to comply the theoretical assumptions described in previous sections. This pendulum is attached to the moving support using cardan joint, see in Fig. 3b. The moving support is made by means of trolley moving on two parallel miniature rails. The kinematic excitation (movement) of the trolley is provided by an electrodynamic shaker. This is shown on the Fig. 3c. The 5500 kN TIRAVib shaker is used to provide the sinusoidal input and is designed to have a 100 mm peak-to-peak maximum displacement. Measurements of the horizontal excitation amplitude are obtained using a LVDT dilatometer screwed on the shaker base, while the motion of the pendulum is measured by a pair of identical high-speed rotary magnetic encoder. It has a non-contact, frictionless design and high speed operation to 30 000 RPM, with 8192 counts per revolution and accuracy 0.5^o. An simple code is developed in MATLAB in order to obtain and store the time histories of the angular position of the pendulum.

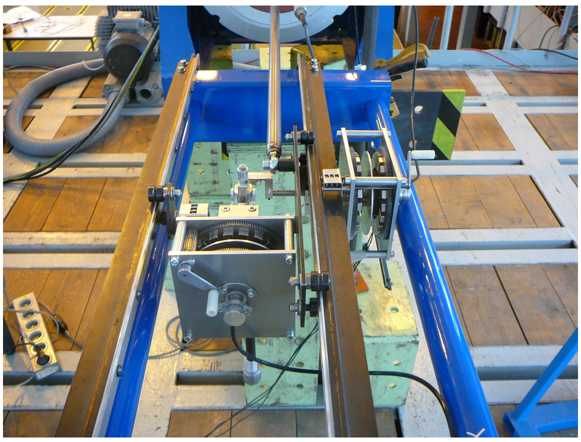
An important part of the experimental gadget with spherical pendulum is a magneto-dynamic unit with the possibility to introduce the viscous damping, i.e. damping force which is a linear function of the motion velocity, for the complete range of amplitudes. It is based on the Lenz's law from the magnetic field theory. It uses the effect of eddy currents induced in the aluminium disk moving between two disks with magnets and acting as a damper. This can be seen on the Fig. 3a and Fig. 3c,d. There is a crank spindle mechanism, which allows to adjust the damping in the practically full range from zero to the critical one.



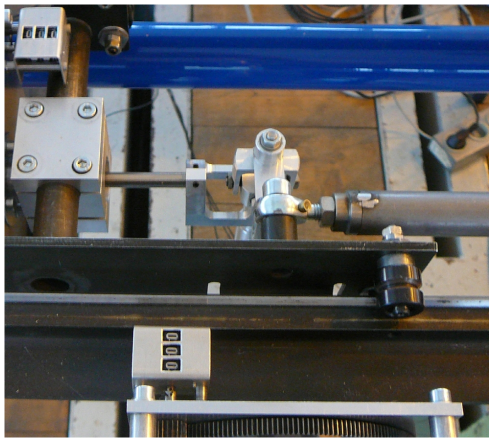
(a) Damping unit with aluminium disk and counter.



(b) DDOF pendulum attached to moving support with cardan joint.



(c) View on the experimental set-up connected to the shaker.

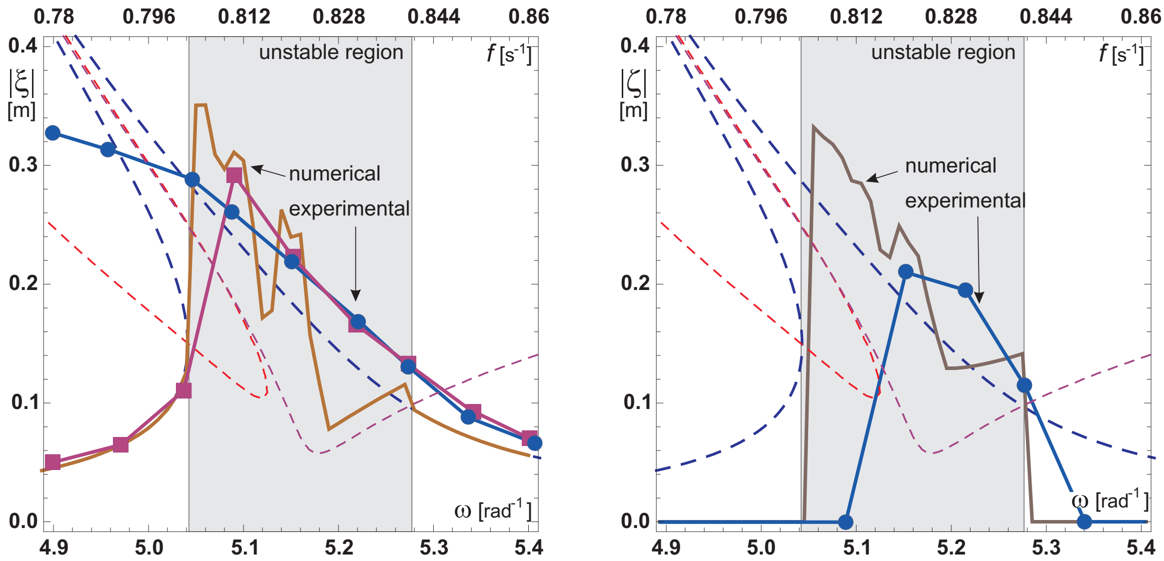


(d) Attachment of cardan joint, pendulum and the trolley with damping units.

Figure 3: Experimental set-up used for the measurements of auto-parametric vibration of kinematically excited spherical pendulum.

Modal parameters of the dynamics system has been measured. The natural frequencies for both directions and the viscous damping coefficients has been determined from the sinus transmission response function and from the free decay of the motion, respectively. The construction of the pendulum differs slightly from the ideal one. Therefore, an equivalent mathematical pendulum with the properties based on the measurement has been taken in the numerical solution.

At this state of the experiments and analysis, the “resonance” curve was investigated changing the frequency and keeping the amplitude constant. In some cases, the pendulum brings itself into an auto-parametric oscillation, while in other cases, the small angular deviation is applied. There were registered several cases of asymptotic stability of semi-trivial solution, as well as the cyclic oscillation with both regular and quasi-harmonic patterns. The forcing frequency has



(a) Numerical and experimental resonance curves for in-plane motion.

(b) Numerical and experimental resonance curves for transversal motion.

Figure 4: Resonance curves for in-plane motion (ξ) and transversal motion (ζ). The experimental values of (ξ) denoted with □ were obtained during sweep-up (for increasing frequency), the curve denoted by ∘ shows amplitudes reached during sweep-down. The dashed lines correspond to the theoretical response curve and stability limits, see Fig. 2.

been taken from $f_l = 0.78$ to $f_u = 0.86$ Hz in 0.01 Hz increments. Damping has been set to the value $\omega_b = 0.0246\,s^{-1}$ for both directions. The natural frequency for both direction is $0.821\,s^{-1}$.

3.2. Results and discussion

Two types of results are presented in this work. The "resonance curves" shown in Fig. 4 and comparison of experimental/numerical response histories, see Figs 5 and 6.

Regarding the resonance curves, there are several noticeable results:

1. The agreement of the numerical, theoretical and experimental values is rather good, especially for the frequencies below the critical region. It is interesting especially when taking into account, that the differential Eq. (7) has been derived using a "small amplitude" assumption.

2. The in-plane motion (ξ variable) shows different stationary amplitudes for "sweep up" and "sweep down" experiments. This fact corresponds well with the both stable branches of the theoretical resonance curve.

3. The transversal motion (ζ variable) of the pendulum should be usually initiated using a small initial impulse, though an "ambient" disturbance sometimes occur. However, in the resonance region this initial impulse is amplified and the pendulum demonstrates the full spatial movement.

The detailed analysis of the several types of the unstable resonance response of the pendulum was published before, see Naprstek (2009). It is worth to emphasise, that all the predicted types of spatial movement has been validated by the experiments with specially designed system. Comparing Figs 5 and 6 one can confirm good qualitative agreement of numerical solution and experimental results, that can be commented as follows:

1. $f = 0.80\,s^{-1}$. The response is stable in the (xz) plane, ζ remains still negligible. However the amplitude is increasing dramatically.

2. $f = 0.81 – 0.82\,s^{-1}$. The response in both components ξ, ζ has a character of beating effects with dominating ξ component and ζ having a character of periodic excursions. The period of these beatings becomes shorter with increasing frequency. The amplitudes don't converge to any constants, they remain quasi-periodical with fluctuating internal structure within individual periods. The superiority of the ξ component follows from the presence of the stability limit ξ in this frequency interval.

3. $f = 0.83 – 0.84\,s^{-1}$. The system response tends to a stationary solution being represented by a space curve resembling the ellipse. For this frequency region is the correspondence of the numerical resonance curve and experimental results not as good as in other cases. The numerical resonance curve follows the ζ stability limit in this frequency region, in opposite to the experimental resonance curve, which roughly follows the original (linear) resonance curve.

4. $f \leq 0.85\,s^{-1}$. The system response is stable again. Possible artificial impulse in the transversal direction diminish in time and finally vanishes.

4. Conclusions

Analytical, experimental and numerical investigation has shown that widely used linear model of the damping pendulum is acceptable only in a very limited extent of parameters concerning pendulum characteristics and excitation properties. Two degrees of freedom non-linear model either in spherical or Cartesian coordinates must be introduced. The harmonic kinematic external excitation in the suspension point has been applied both analytically and experimentally.

There are three types of the resonance domains. They can be identified with respect to relation of a preliminary resonance curve and stability limits. If the stability limits intersected the resonance curve, then four different response regimes can be consecutively encountered in the resonance domain. For large excitation amplitude and/or small damping can the instability interval occupy a large frequency domain. It implies a design recommendation that the damping should not drop bellow a certain limit to avoid the instability which implies the decrease of the pendulum efficiency especially for broad band excitations.

From the practical point of view, there is highly recommended to design the damping pendulum in such a way that any intersections of the resonance curve with the stability limits are avoided. Especially intersection with in-plane (ξ) stability limit should be prevented, otherwise negative influence of the pendulum in the resonance domain is to be expected in both along-wind as well as in cross-wind directions.

The experiments restricted themselves to one excitation amplitude and the single value of damping and more experiments are envisaged. They will focus upon the influence of damping and the amplitude of excitation on the semi-trivial stability.

Acknowledgements

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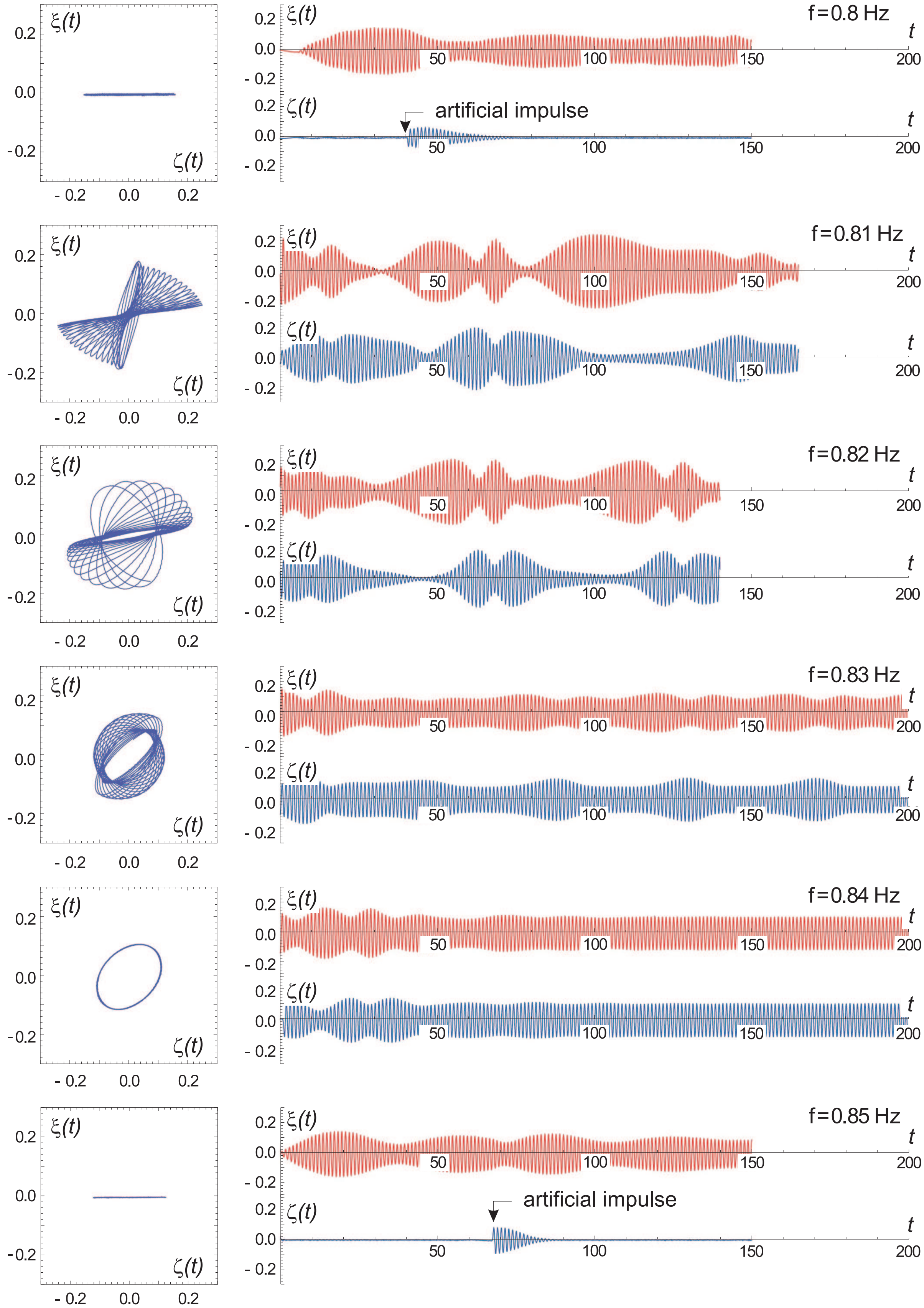

Figure 5: Results of the experimental measurement. For each frequency, the time histories of $\xi(t)$ (in plane) and $\zeta(t)$ (transversal) motion are depicted in the right part. The plots $\xi(t)$ vs. $\zeta(t)$ are on the left side. Units are [s] for time and [m] for displacement.

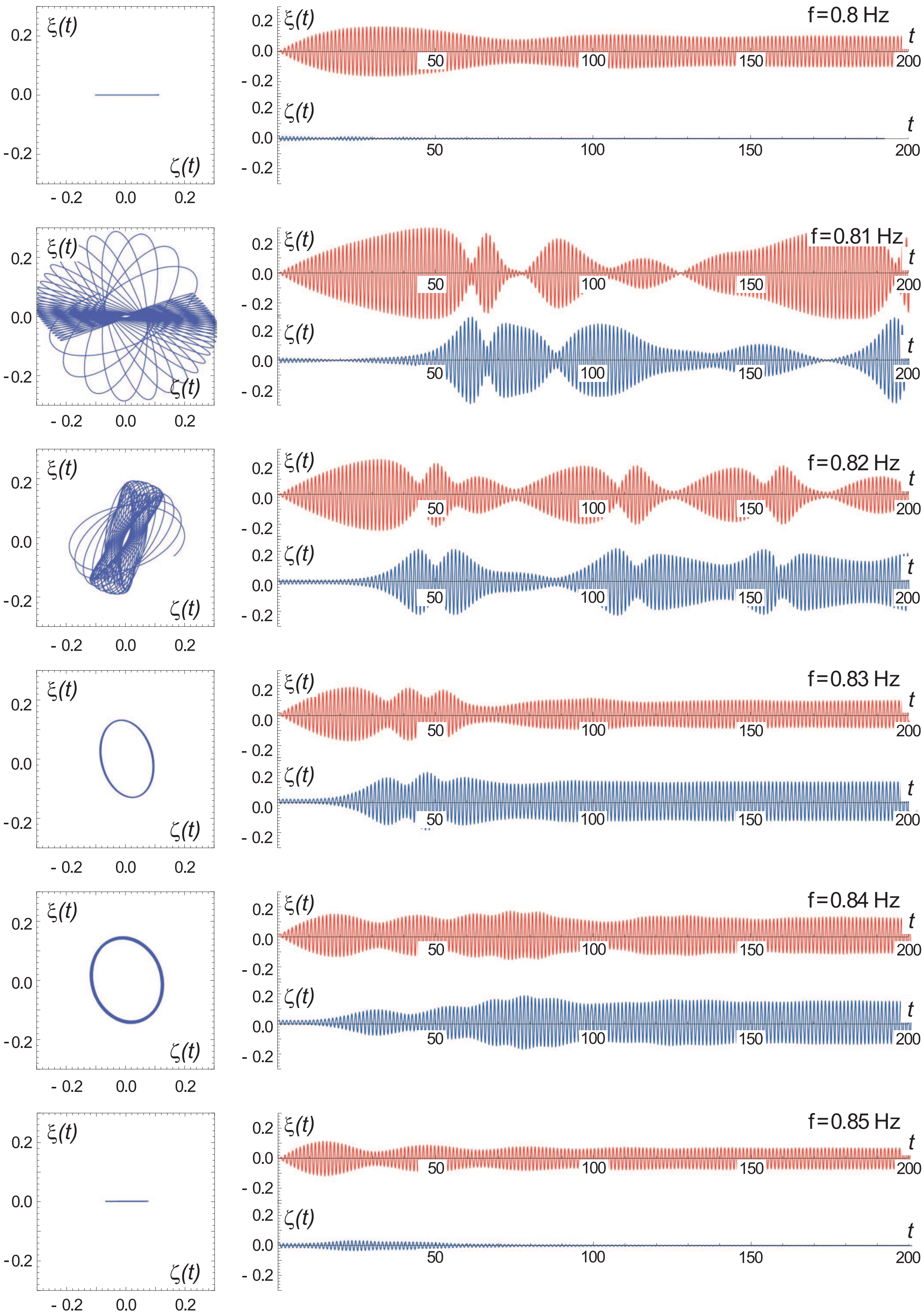

Figure 6: Results of the numerical integration of the approximate system of Eqs (2). For each frequency, the time histories of $\xi(t)$ (in plane) and $\zeta(t)$ (transversal) motion are depicted in the right part. The plots $\xi(t)$ vs. $\zeta(t)$ are on the left side. Units are [s] for time and [m] for displacement.