ALTERNATIVE EXPRESSION FOR STABLE AND UNSTABLE LIMIT CYCLES OF AERO-ELASTIC PENDULUM

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Summary: *The subject considered in this paper is the aero-elastic self-induced oscillation of a mechanical system with generalized single degree-of-freedom (SDOF). The motion is described by an ordinary differential equation of Duffing type with special form of aero-elastic damping terms of Van der Pol type. A new semi-analytical approach is introduced in order to identify the limit cycles, both stable as well as unstable, which are not possible to be identified by means of an experiments nor by a numerical integration.*

1. Introduction

The vibration and the stability of a prismatic body in an air flow is often a result of the aero-elastic interaction of the response and the forces varying in time which have the non-conservative and gyroscopic nature. Several types of aero-elastic oscillation are known using technical language in the wind engineering: flutter, galloping or self-excited motion induced by separating vortices at the body or in the wake of it. Each of them can be observed separately depending on the geometry and mechanical properties of the structure and the flow conditions. Very often however, the distinction may not be so clear. A bluff body structure can display both flutter and galloping characteristics, whereas each of them may be also affected by the wind turbulence. For example, during the life of the Tacoma Narrows bridge the experience with the vertical and torsional vibration had been associated with the galloping or torsional flutter respectively, see e.g. Billah & Scanlan (1991). General view at the conditions of dynamic stability and physical interpretation is given in Náprstek (2007), where several types of aero-elastic stability loss known from engineering practice and experimental data are identified in the frequency × frequency × air stream velocity domain.

During the past decades, many explanations of the aero-elastic phenomenon and many mathematical models have been suggested. However all of them are suffering from uncertainty regarding the role of individual parameters being fundamental for the instability origin in a particular technical branch. Moreover, the analysis has revealed relatively considerable diversity of conclusions following from experimental studies. This is probably due to historical treatment of these problems in a number of branches as well as due to the existence of various instability domains and of a number of bifurcation points types. In the course of time, however, the research has succeeded in understanding that the majority of the models have either obvious or hidden linear character being based on various types of convolution formulations. Although some of these approaches are able to predict lower limits of aero-elastic stability, they do not provide any

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possibility to investigate the post-critical behavior, which is of strongly non-linear character. In any case, the detailed knowledge of the post-critical state is very important being decisive from the viewpoint of a possible secondary re-stabilization due to non-linear effects.

There are many papers dealing with just the single degree-of-freedom aero-elastic system starting with the paper by Parkinson & Smith (1963). This paper deals with the quasi-steady analysis of transverse galloping of a long square prism in a normal steady wind. The aerodynamic force coefficient is modeled as the polynomial of the seventh degree and the respective equation is solved by Krylov and Bogolyubov method. More recently, other paper appeared. In Oudheusden (2000) for example, galloping oscillation with a single rotational degree-of-freedom is investigated with the unsteady aerodynamics forces taken as a combination of aerodynamic stiffness and damping terms. In the paper by Leech (1979) the aerodynamic yaw moment is written in a form of Taylor-Fourier expansion and then a mathematical model similar to the Van der Pol equation is created. The stable and instable limit cycles are discussed. The paper by Matsumoto (1997) describes the torsional flutter mechanism of 2D rectangular cylinders and 2D H-shaped cylinders based upon unsteady pressure measurements under forced torsional vibration. For cylinders with fixed side ratio in the high reduced velocity range, the torsional flutter mechanism is in principle identical with that of coupled flutter. However, in the low reduced velocity range the torsional flutter is induced by vortices at along side-surface of the cylinder and thus differs from mechanism of coupled flutter. In the paper by Vio et al. (2007), the square cross-section beam in a normal steady flow is analysed with regards to the global stability conditions. Several methods are used in this paper in order to predict bifurcation points. Comparison with numerical integration is given. Luongo et al. (2008) evaluated the branches of periodic solutions and their stability as functions of wind velocity. The existence of quasi-periodic solutions is proved.

Generally speaking, however, some types of non-linear effects can act like stabilizing factors. They are able at least for a limited period of time, to restitute one of lower types of stability after the structure has lost its exponential or asymptotic stability, see Tondl (1999). In Lumbantobing & Haaker (2004) the model of the parametric excitation of the aero-elastic oscillators is considered in a form of Mathieu nonlinear equation. The possibility of re-stabilization of the trivial solution after up-crossing the critical state is shown.

The subject considered in this paper is the non-linear dynamics of a mechanical oscillator with one degree of freedom in a wind field arising from reduction of the model described in Náprstek et al. (2008). The schematic picture of the system can be viewed at the Fig. 1.
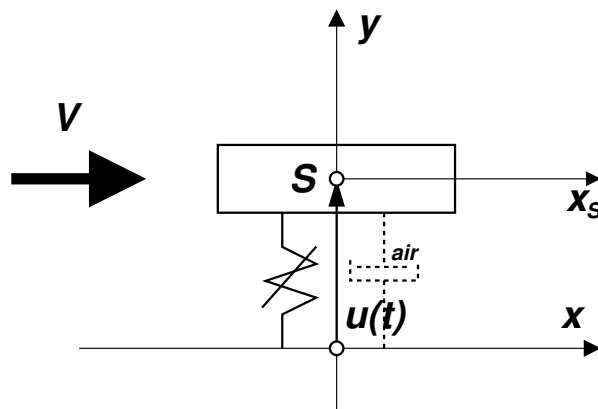

Figure 1. Model of the self-excited aero-elastic SDOF structure with nonlinear stiffness and aero-elastic damping.

It has been shown that such system has several types of bifurcation manifolds. The conditions of existence and relevant portraits of the principal limit cycles have been carried out. In this paper, the new approach is introduced. The gyroscopic forces are written in a form of the polynomial of the second and consequently of the fourth order in order to assume the possibility of occurring also a non-stable limit cycles, which are not possible to be identified by means of experiments or numerical integration.

2.   Problem outline

The relevant equation can be written in a form:

$$\ddot{u} + g(u) = \mu f(u, \dot{u}) \tag{1}$$

The right hand side of the equation, the aerodynamic forcing function $f(u, \dot{u})$ is depending on the geometry and the wind speed. Generalizing the common harmonic assumption for the response, we assume the solution in a form

$$u(t) = a \cos \varphi(t) + b \tag{2}$$

In this formula, the generalized phase $\varphi(t)$ is used. It can be naturally explained as a function given by a relation $d\varphi(t)/dt = \Phi(\varphi)$. The generalized frequency $\Phi(\varphi)$ is the periodic function with the period 2π. The coefficient a is the amplitude of the function and the coefficient b determines the eccentricity of the response with respect to the origin. Using the chain differentiation rule, we may rewrite Eq. (1) in the following form:

$$\Phi \cdot \frac{d}{d\varphi}(\Phi u') + g(u) = \mu f(u, \Phi u') \tag{3}$$

Eq. (3) represents a transform of Eq. (1) into the coordinate $\varphi(t)$. The both sides of the Eq. (3) should be multiplied by $u' = -a \sin \varphi$ and integrated across the interval $\widetilde{\varphi} \in \langle 0, \varphi \rangle$. This leads to the formula:

$$\int_0^\varphi \frac{1}{2}(\frac{d}{d\widetilde{\varphi}}(\Phi u')^2)d\varphi + \int_0^\varphi g(u)du = -\mu a \int_0^\varphi f(u, \Phi u') \sin \widetilde{\varphi}d\widetilde{\varphi} \tag{4}$$

Hence the fundamental expression, which enables to deduce the resulting equation for the establishment of the limit cycles is obtained:

$$\frac{1}{2}(\Phi a \sin \varphi)^2 + h(a \cos \varphi + b) - h(a + b) = -\mu a \int_0^\varphi f\left(a \cos \widetilde{\varphi} + b, -a\Phi \sin \widetilde{\varphi}\right) \sin \widetilde{\varphi} \, d\widetilde{\varphi} \tag{5}$$

The function h in this formula represents the indefinite integral of $g(u)$ i.e. $h(u) = \int g(u)\, du$. Evaluating Eq. (5) for: (i) $\widetilde{\varphi} \in \langle 0, 2\pi \rangle$ and (ii) $\widetilde{\varphi} \in \langle 0, \pi \rangle$, two important conditions can be deduced respectively:

$$\begin{aligned}
\int_0^{2\pi} f\left(a_k, b_k, \varphi, \Phi_k\right) \sin \varphi \, d\varphi &= 0 \quad (i) \\
h(a_k + b_k) - h(-a_k + b_k) - \mu a_k \int_0^\pi f\left(a_k, b_k, \varphi, \Phi_{k-1}\right) \sin \varphi \, d\varphi &= 0 \quad (ii)
\end{aligned} \tag{6}$$

Eqs. (6) allow to calculate values of a and b and approximation of $\Phi(\varphi)$ and hence the solution of non-linear equation in an iterative loop. This approximation is converging rapidly to the exact solution (calculated for example numerically). The balance has to be calculated after each half of the cycle due to possibility that $b \neq 0$ in case that the characteristics are not symmetric with respect to the origin. The period of the limit cycle is given by the formula

$$T = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\Phi(\varphi)} \tag{7}$$

2.1. The first approximation

Often, the first approximation of the solution is important especially when it can be also calculated analytically. In this case, if the parameter $\mu \approx 0$, the first approximation of $\Phi_k = \Phi_0$, corresponding to the solution of homogeneous form of the Eq. (1) can be obtained, i.e.:

$$\Phi_0 = \left[\frac{2h(a_0 + b_0) - 2h(a_0 \cos \varphi + b_0)}{a_0^2 \sin^2 \varphi} \right]^{1/2} \tag{8}$$

Employing the right hand side with nonlinear damping, it can be relatively easy seen, that for the $k-$th approximation with $\mu > 0$ one may obtain the higher approximations of Φ:

$$\Phi_k(\varphi) = \left[\frac{2h(a_k + b_k) - 2h(a_k \cos \varphi + b_k) - 2\mu a_k \int_0^\varphi f\left(a_k, b_k, \widetilde{\varphi}, \Phi_{k-1}\right) \sin \widetilde{\varphi}\, \mathrm{d}\widetilde{\varphi}}{a_k^2 \sin^2 \varphi} \right]^{1/2} \tag{9}$$

Knowing the generalized frequency, we are able to establish the stability of the periodic solution. According to the Floquet theory, it can be determined by an exponent λ, called multiplier of the periodic orbit which is characterising the phase volume evolution. It is given as the trace of Jacobi matrix of the system (1), i.e. by the formula:

$$\lambda = \int_0^T \frac{\mathrm{d}f(u, \dot{u})}{\mathrm{d}\dot{u}}\, \mathrm{d}t \tag{10}$$

When λ has negative value, the limit cycle is stable, whereas for λ positive, the limit cycle is instable. If trajectory is stable according to Lyapunov, then arbitrary initial perturbation doesn't prove any grow, on average, along the trajectory.

3. Example-stable limit cycle

Let us demonstrate this approach on the equation describing the motion of the system with one degree-of-freedom (heave or rotation) oscillating in the flow, possibly reaching limit cycles oscillation. Firstly, we write the right hand side in the form of Van der Pol damping:

$$\ddot{u} + \alpha u + \beta u^3 = \mu(\eta - \nu u^2)\dot{u} \tag{11}$$

where α, β, ν, η and μ are the equation coefficient which can have different values. Applying the Eq. (8) we obtain the first approximation of the generalized frequency $\Phi(\varphi)$ in an analytical form. It holds with the respect to Eq. (8):

$$\Phi_0 = [\alpha + \frac{1}{2}\beta a_0^2(1 + \cos^2\varphi)]^{\frac{1}{2}} \tag{12}$$

Integrating this formula, the generalized phase as well as the flow field in the plane $(u, \dot{u})$ can be calculated. Finally, the coefficient λ of the first approximation may be calculated by means of Eq (10) as:

$$\lambda = \frac{2\nu}{\beta} \int_0^{2\pi} \frac{\varrho - a_0^2\cos^2\varphi}{(\delta + a_0^2(1 + \cos^2\varphi))^{\frac{1}{2}}}\mathrm{d}\varphi; \quad \varrho = \frac{\eta}{\nu}; \quad \delta = \frac{2\alpha}{\beta} \tag{13}$$

The λ coefficient depends strongly on the combination of the parameters ϱ and δ. It should be less than zero in order to observe the stable limit as said before. The calculated limit cycle for the $\delta = 2$ and $\varrho = 1$ together with the amplitude of vibration is given at the Fig. 2.
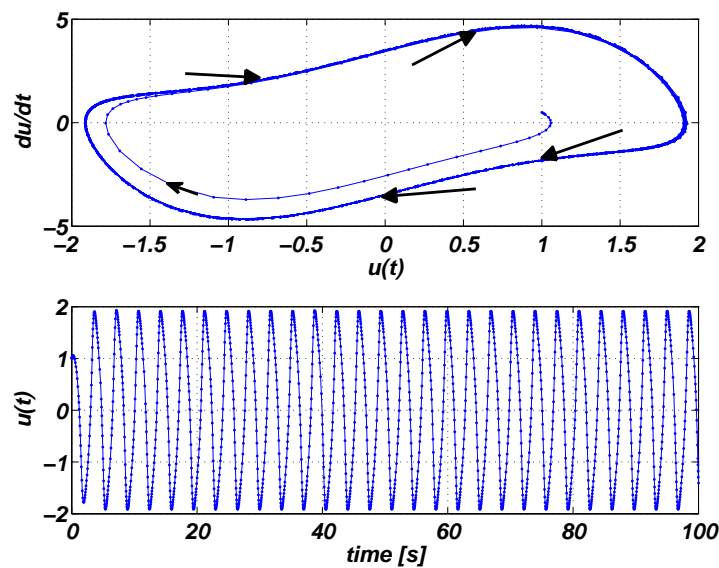

Figure 2. Available numerical solution of the Eq. (11) with stable limit cycles. The parameters values: $\alpha = 1$, $\beta = 1$, $\nu = 1$, $\eta = 1$ and $\mu = 2$.

4.   Example-unstable limit cycle

By extension of the right hand side of the Eq. (11) we may demonstrate the behaviour of the equation with the possibility of the existence of unstable limit cycles. The following adoption is carried out and discussed:

$$\ddot{u} + \alpha u + \beta u^3 = \mu(\eta - \nu u^2 + \vartheta u^4)\dot{u} \tag{14}$$

It should be particularly noticed, that the right hand side of Eq. (14) includes the fourth degree of the response in order to encompass possibly both stable and unstable limit cycles as it has been observed also experimentally in a wind channel, see Náprstek et al. (2008). Their existence is predetermined by a particular ratio of parameters η, ν, ϑ. The theoretical solution of the above equation shows the considerable sensitivity of the system self-excited vibration with respect to particular values of parameters. The structure of the right-hand side of the governing

equation leads to the oscillation around the origin $(0, 0)$, thus the value of b is zero. Including the odd terms on the right hand side of this equation, however, would lead to the shift of the origin in the phase plane and consequently to the nonzero value of b.
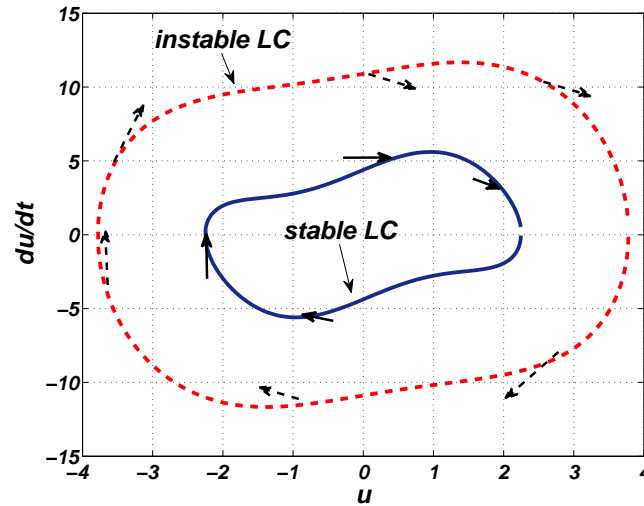

Figure 3. Attractive and repulsive limit cycles as a solution of Eq. (14). The parameters values: $\alpha = 1, \beta = 1, \nu = 1, \eta = 1$ and $\mu = 2$. The parameter ϑ is varying in the interval $\vartheta \in \langle -0.2, 0.2 \rangle$.

The advantage of the proposed method is demonstrated, while only stable limit cycles, i,e. the cases when the bifurcation parameter $\vartheta < 1$, can be found by a common numerical procedure, this method is able to depict the unstable limit cycles as it is shown at the Fig. (3. The stability diagram complementary to the graphs of limit cycles is given on the Fig. 4, where the bifurcation diagram valid for certain μ, η and ν is shown.
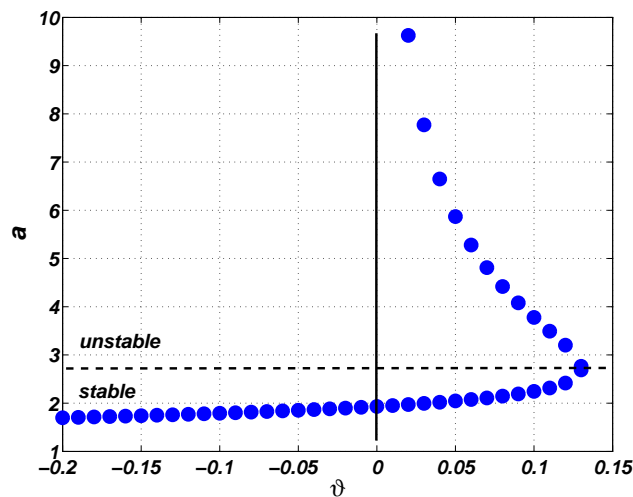

Figure 4. Stability diagram for the parameter ϑ. The parameters values: $\alpha = 1, \beta = 1, \nu = 1, \eta = 1$ and $\mu = 2$.

Let us look at the case when the stiffness terms, the function $g(u) = \alpha u + \beta u^3$, vary. Two cases are shown. Firstly, it has only on ezero point at the point $u = 0$, i.e. $\alpha > 0, \beta > 0$. Such case is shown on the Fig. 5. On the Fig. 6 the case when the function is crossing the axis u not only at the origin. This means that for example: $\alpha < 0, \beta > 0$. The stability index λ vary

with the parameter ϑ and consequently does the generalized amplitude a with λ, as implies the Eq. (13).
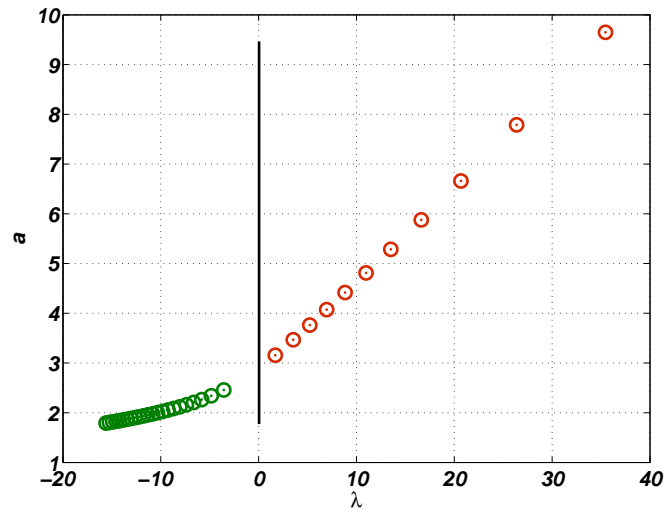

Figure 5. The amplitude a as a function of λ. The parameters values: $\alpha = -0.1$, $\beta = 0.1$, $\nu = 1$, $\eta = 1$ and $\mu = 2$.
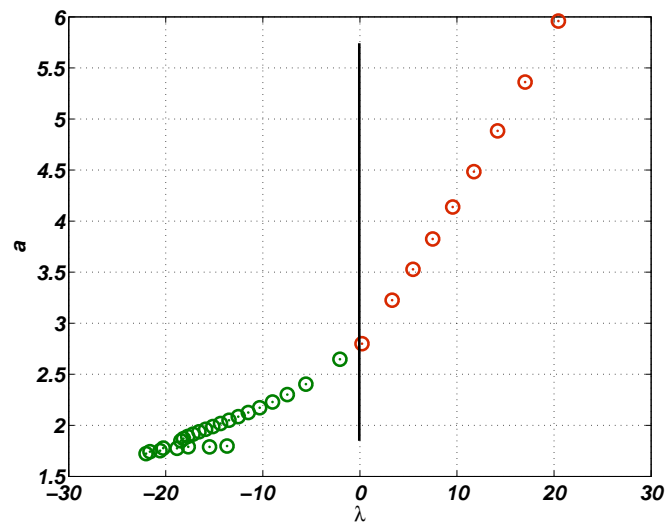

Figure 6. The amplitude a as a function of λ. The parameters values: $\alpha = -0.1$, $\beta = 0.1$, $\nu = 1$, $\eta = 1$ and $\mu = 2$.

5.   Conclusions

After the loss of trivial solution stability the response in one degree of freedom tends to stabilize itself in the form of a stable limit cycle, see Fig. 3. The solid curve represents an attractor for all configurations of initial conditions, when parameter $\vartheta < 0$, see Fig. 4. For $\vartheta > 0$ stable ($\lambda < 0$) and unstable ($\lambda > 0$) limit cycles exist until the point ($\vartheta = \vartheta_{max}$, $\lambda = 0$) is reached. This point provides a twofold limit cycle and represents a strong energy barrier. When this energy barrier is overcame by means of further wind energy supply, for example, the limit cycles don't exist any more and the final stability loss occurs. The system response starts to

grow beyond all limits. This process can be initiated from the viewpoint of our analysis when a is increasing and inducing a transition of λ through the zero value. The twofold limit cycle making an important separatrix manifold cannot be directly determined neither by means of experiments nor the numerical integration.

6. Acknowledgment

7. References

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