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FORWARD KINEMATICAL SOLUTION OF HEXASPHERE BY STRUCTURAL APPROXIMATION

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Summary: *The paper deals with the description of a new method for the kinematical solution of non-simple (analytically non-solvable) mechanisms. The computational complexity of this method is better than the traditional iterative Newton method because it does not require the assembly of Jacobi matrix and the Gauss elimination. The method is based on the removal of structural part of the mechanism being analysed that causes the analytical non-solvability. The method is applied for the forward kinematical solution of HexaSphere.*

1. Introduction

The investigated problem is the solution of positional kinematical problem for analytically non-solvable (so called non-simple) mechanical systems (mechanisms) (Stejskal & Valasek 1996). The traditional solution method is the Newton method. However, this paper deals with a new method for positional kinematical solution of mechanisms with loops. The method is based on the concept of structural approximation, i.e. the structure of the mechanism being solved is simplified in such a way that the mechanism with simplified structure is analytically solvable. The analytical solution is the basis of the iteration. This method has been successfully applied for the inverse kinematical solution of non-simple serial robots (Kalny & Valasek 1991). This paper extends this method for mechanisms with loops and specifically for forward kinematical solution of parallel kinematical structures. The method of structural approximation is demonstrated on HexaSphere – a new concept of redundant parallel mechanism.

2. Method of Structural Approximation

If a kinematical structure is not analytically solvable then it includes usually some structural (topological) feature that is responsible for this non-solvability. If this feature is removed the resulting kinematical structure becomes solvable (Fig. 1). Such feature is for example the distance of rotational axes in non-spherical robot wrist (Stejskal & Valasek 1996). If this distance is set to zero, the serial robot becomes simple and the inverse kinematical problem is solvable. This analytical solution is then computed for the perturbed right-hand side of kinematics constraints evaluated from the pervious values of coordinates. This is the basis for iterations.

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The kinematical structure is described by the coordinates $\mathbf{s}$. These coordinates are constrained by the kinematical constraints:

$$\mathbf{f}(\mathbf{s}) = \mathbf{0} \tag{1}$$

These equations are not analytically solvable. But they can be split into the simple part $\mathbf{f}_S$ that is analytically solvable and the non-simple part $\mathbf{f}_{NS}$ that is causing the non-solvability. This corresponds to the Fig. 1.

$$\mathbf{f}(\mathbf{s}) = \mathbf{f}_S(\mathbf{s}) + \mathbf{f}_{NS}(\mathbf{s}) = \mathbf{0} \tag{2}$$

Because the part $\mathbf{f}_S$ is analytically solvable it can be developed an iteration scheme:

$$\mathbf{f}_S(\mathbf{s}) = -\mathbf{f}_{NS}(\mathbf{s}) \tag{3}$$

$$\mathbf{s}_{i+1} = \mathbf{f}_S^{-1}(\mathbf{f}_{NS}(\mathbf{s}_i)) \tag{4}$$

This iteration scheme converges and this can be checked by traditional means (Rektorys 1989) of comparison of magnitudes of partial derivatives on left and right hand sides of (3).

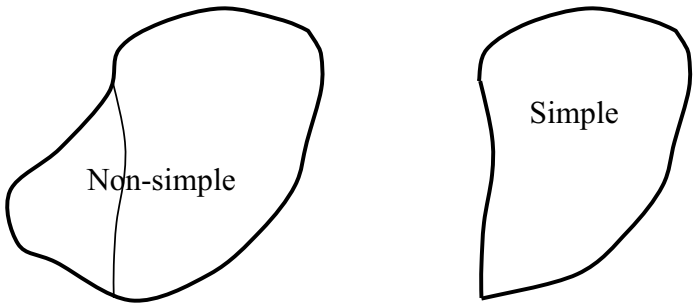

Figure 1 Non-simple kinematical structure and its structural approximation

3. HexaSphere

HexaSphere is a new concept of redundant parallel mechanism for spherical motion with increased workspace (Fig. 2). Its purpose is to orient the platform and during that keep the point in the middle of the platform on the spherical surface. It was designed in the Department of Mechanics, Biomechanics and Mechatronics. It was successfully presented on the MSV 2008 in Brno.

The platform position can be described by 3 Euler angles – precession ψ, nutation ϑ and rotation φ. To avoid collisions of the legs it has to be fulfilled $\varphi = -\psi$ then the platform is rolling during the motion. The workspace is $\psi = \langle 0°, 360° \rangle$, $\vartheta = \langle 0°, 100° \rangle$. There is not also any singular cases in the workspace. Instead of that it has to be dealt with the fact it is needed to control 6 translational positions s_i to orient the platform with only 3 degrees of freedom. That is the fundament of redundant mechanisms and also their biggest issue.

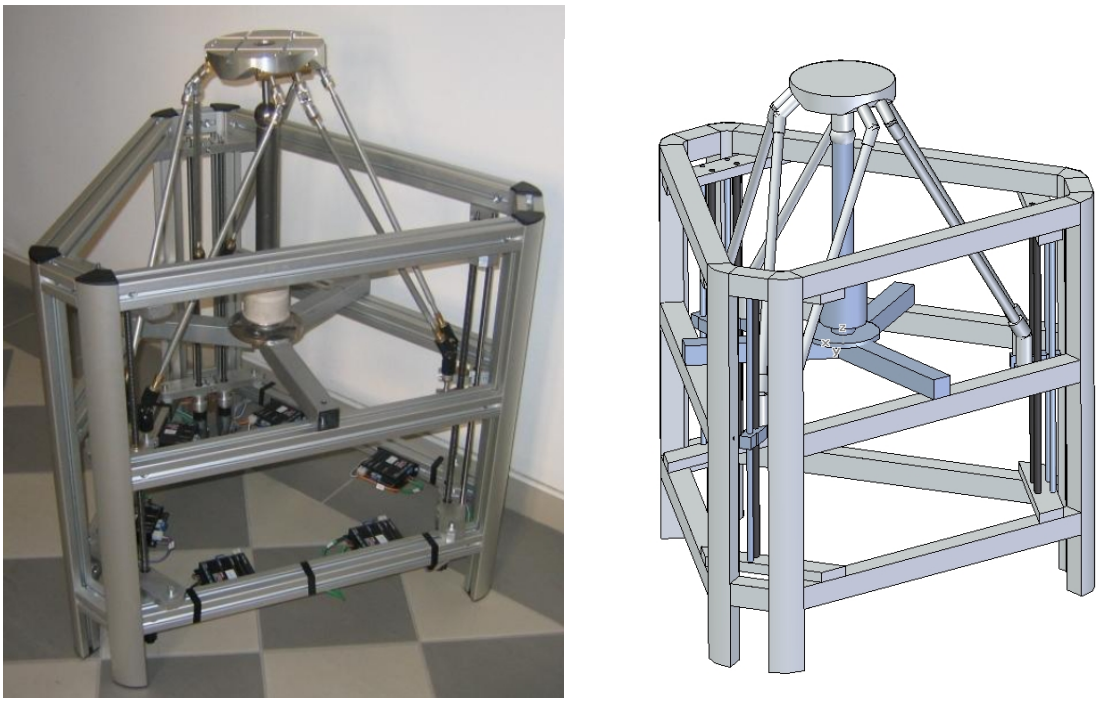

Figure 2 HexaSphere – prototype and CAD model

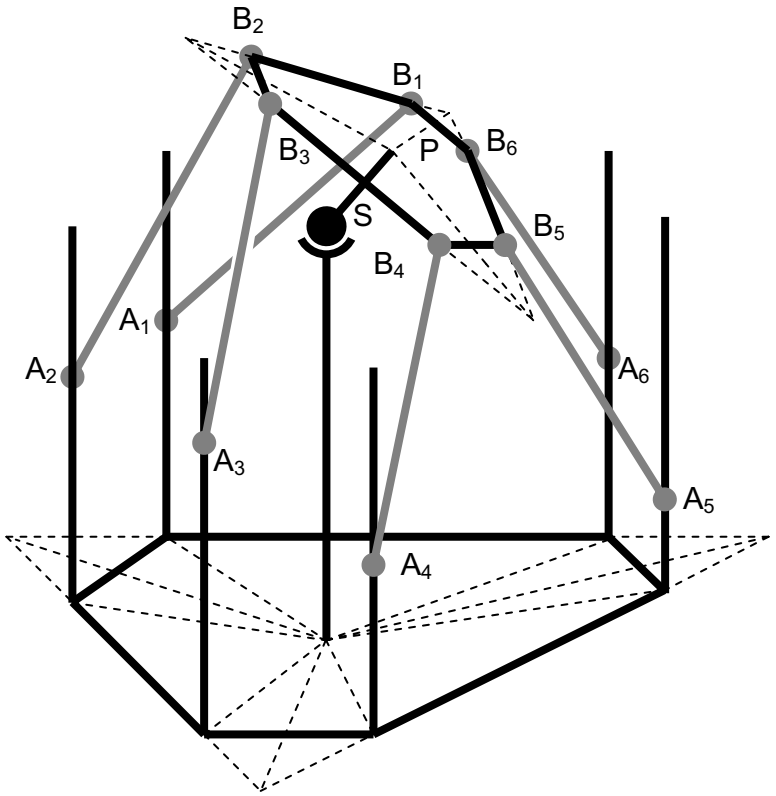

Figure 3 HexaSphere – kinematical model

4. Forward Kinematical Solution of HexaSphere by Newton iterative method

The traditional approach for solving the kinematical problems is Newton iterative method. Mathematically, it is solving of a set of several nonlinear equations (1). The method is based on the Taylor expansion (1) to the first order in the neighbourhood of initial value $\mathbf{s}^{(k)}$

$$\mathbf{f}(\mathbf{s}) = \mathbf{f}\left(\mathbf{s}^{(k)}\right) + \frac{\partial \mathbf{f}}{\partial \mathbf{s}^T}\Delta\mathbf{s}^{(k)} = \mathbf{0} \tag{5}$$

where

$$\mathbf{\Phi}(\mathbf{s}) = \frac{\partial \mathbf{f}(\mathbf{s})}{\partial \mathbf{s}^T} \tag{6}$$

is a so-called Jacobian. From (5) we obtain a system of linear equation for $\Delta\mathbf{s}^{(k)}$:

$$\mathbf{\Phi}(\mathbf{s})\Delta\mathbf{s}^{(k)} = -\mathbf{f}\left(\mathbf{s}^{(k)}\right) \tag{7}$$

If the Jacobian is a non-singular matrix, it is possible to solve the system

$$\Delta\mathbf{s}^{(k)} = -\mathbf{\Phi}^{-1}\mathbf{f}\left(\mathbf{s}^{(k)}\right) \tag{8}$$

$$\mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} + \Delta\mathbf{s}^{(k)} \tag{9}$$

The number of independent loops is 6 (= *number of joints – number of* + 1). Therefore, when solving the forward kinematical problem of HexaSphere we have to fulfilled 6 nonlinear equations but only in 2 unknowns – precession angle ψ and nutation angle . The Jacobian is not square matrix and we have to solve the overconstrained system. This is typical especially for redundant mechanisms. The exact solution generally does not exist. Instead of this only the solution with minimum residui can be found. However, such a procedure is quite computationally expensive.

5. Forward Kinematical Solution of HexaSphere by Structural Approximation

The forward kinematical problem of HexaSphere, that is not analytically solvable, has been solved by the method of structural approximation.

The reason why the HexaSphere is not analytically solvable are non-zero distances $\left|\overline{B_1B_6}\right|$, $\left|\overline{B_2B_3}\right|$, $\left|\overline{B_4B_5}\right|$. The design issues are the only but insurmountable reason of these distances. However, if we set them to zero, the platform will become triangular and the position of its vertices is analytically solvable – they are determined as the intersection of three spherical surfaces from points A_i, A'_{i+1}, S with constant radiuses. This is the simple part of the mechanism.

In case of the real HexaSphere we have to approximate it by such a simple part. Although, there are six significant points B_i on the platform, the position of the platform in space can be expressed only by 3 points which create a triangle. Let us use points B_2, B_4, B_6. For example, the structural approximation for solving the position of point B_4 is introduced by setting the distance B_4B_5 to zero.

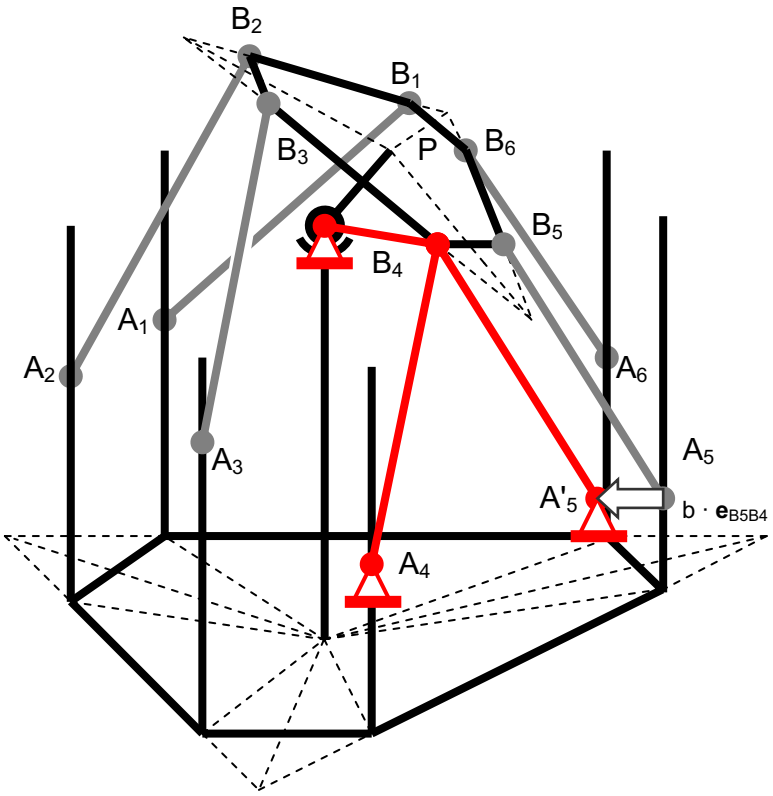

Figure 4 Structural approximation of hexapod for computing B4 position

However, the particular structural approximation is carried out by supposing the knowledge of the vector B_4B_5 from the previous solution (iteration). The point A_5 is moved by B_5B_4 to the point A'_5. The resulting mechanism is analytically solvable (Fig. 4). The position of the point B4 is determined as the intersection of three spherical surfaces from points A_4, A'_5 and S.

This can be written as:

$$\left|\mathbf{r}_{B4} - \mathbf{r}_{A4}\right| = L \tag{10}$$

$$\left|\mathbf{r}_{B5} - \mathbf{r}_{A5}\right| = L \tag{11}$$

$$\left|\mathbf{r}_{B6} - \mathbf{r}_{S}\right| = \left|\overline{B_iS}\right| = konst. \tag{12}$$

Structural approximation is expressed as

$$\left|\mathbf{r}_{B4} + \mathbf{r}_{B4B5} - \mathbf{r}_{A5}\right| = L \tag{13}$$

$$\left|\mathbf{r}_{B4} - \left(\mathbf{r}_{A5} + b \cdot \mathbf{e}_{B5B4}\right)\right| = L \tag{14}$$

$$\left|\mathbf{r}_{B4} - \mathbf{r}_{SAP}\right| = L \tag{15}$$

where $\mathbf{r}_{SAP}$ is a vector of the structural approximation, i.e. the radiusvector of the point A'_5.

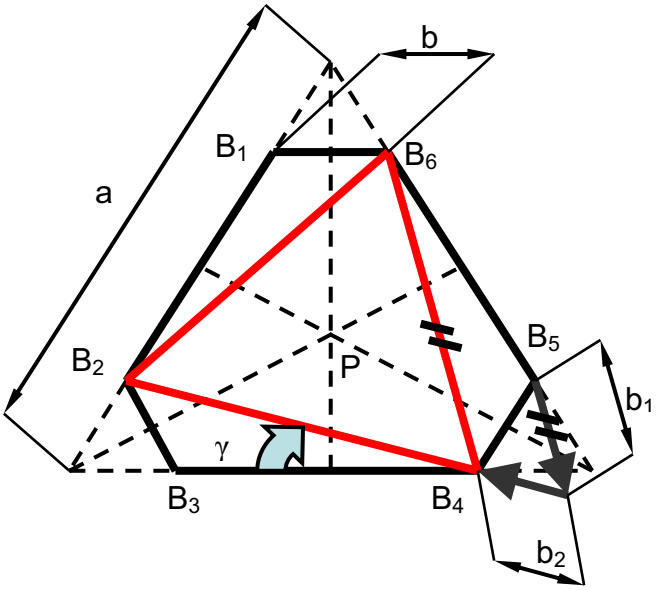

Figure 5 Moving platform

Unfortunately, it is not possible to determined $\mathbf{r}_{SAP} = \mathbf{r}_{A5} + b \cdot \mathbf{e}_{B5B4}$. Instead of, the vector of structural approximation can be expressed as (see Fig. 5):

$$\mathbf{r}_{SAP} = \mathbf{r}_{A5} + b_1 \cdot \mathbf{e}_{B6B4} + b_2 \cdot \mathbf{e}_{B4B2} = \mathbf{r}_{A'5} \tag{16}$$

where b_1 and b_2 are constant. These equations give the intersection of three spherical surfaces:

$$\left(x_{B4} - x_{SAP}\right)^2 + \left(y_{B4} - y_{SAP}\right)^2 + \left(z_{B4} - z_{SAP}\right)^2 = L^2 \tag{17}$$

$$\left(x_{B4} - x_{A4}\right)^2 + \left(y_{B4} - y_{A4}\right)^2 + \left(z_{B4} - z_{A4}\right)^2 = L^2 \tag{18}$$

$$\left(x_{B4} - x_S\right)^2 + \left(y_{B4} - y_S\right)^2 + \left(z_{B4} - z_S\right)^2 = \left|\overline{B_4S}\right|^2 \tag{19}$$

where we changed the vector description over the scalar one whereas

$$\mathbf{r}_{Ai} = \begin{bmatrix} x_{Ai} \\ y_{Ai} \\ z_{Ai} \end{bmatrix}, \ \mathbf{r}_{Bi} = \begin{bmatrix} x_{Bi} \\ y_{Bi} \\ z_{Bi} \end{bmatrix}, \text{ etc.} \tag{20}$$

After multiplication, the third equation (19) can be subtracted from the first (17) and second one (18):

$$\begin{aligned} &2\left(x_S - x_{SAP}\right)x_{B4} + 2\left(y_S - y_{SAP}\right)y_{B4} + 2\left(z_S - z_{SAP}\right)z_{B4} = \ldots \\ &\quad \ldots = L^2 - x_{SAP}^2 - y_{SAP}^2 - z_{SAP}^2 - \left|\overline{B_4S}\right|^2 + x_S^2 + y_S^2 + z_S^2 \end{aligned} \tag{21}$$

$$\begin{aligned} &2\left(x_S - x_{A4}\right)x_{B4} + 2\left(y_S - y_{A4}\right)y_{B4} + 2\left(z_S - z_{A4}\right)z_{B4} = \ldots \\ &\quad \ldots = L^2 - x_{A4}^2 - y_{A4}^2 - z_{A4}^2 - \left|\overline{B_4S}\right|^2 + x_S^2 + y_S^2 + z_S^2 \end{aligned} \tag{22}$$

Equations (21, 22) can be solved as a set of linear equation in unknowns x_{B4}, y_{B4}. Using the Cramer's Rule the solution is linear in a variable z_{B4}:

$$\begin{aligned} x_{B4} &= k_x \cdot z_{B4} + q_x \\ y_{B4} &= k_y \cdot z_{B4} + q_y \end{aligned} \tag{23}$$

By insertion of the solution (23) in the equation (19) we get a quadratic equation for z_{B4}:

$$\begin{aligned} &\underbrace{\left(k_x^2 + k_y^2 + 1\right)}_{\equiv a_z} z_{B4}^2 + \ldots \\ &\underbrace{2\left(k_x q_x - x_S k_x + k_y q_y - y_S k_y - 2 z_S\right)}_{\equiv b_z} z_{B4} + \ldots \\ &\underbrace{q_x^2 + q_y^2 - 2\left(x_S q_x + y_S q_y\right) + x_S^2 + y_S^2 + z_S^2 - \left|\overline{B_4 S}\right|^2}_{\equiv c_z} = 0 \end{aligned} \tag{24}$$

This equation has 2 solutions. The choice is not trivial because both of them are in the work-space of the mechanism. For the majority of positions, especially the upper half-space, the correct solution is always the maximum:

$$\left(z_{Bi}\right)_{1,2} = \frac{-b_z \pm \sqrt{b_z^2 - 4 a_z c_z}}{2 a_z}, \quad z_{Bi} = \max\left[\left(z_{Bi}\right)_1, \left(z_{Bi}\right)_2\right] \tag{25}$$

and the remaining coordinates can be computed from (23).This is repeated for each point B_2, B_4, B_6 of the platform. Then the knowledge of these points is used for the computation of the orientation of the platform and hence again the position of the vector A'_5. It is very essential that the orientation is determined only as the elements of the matrix of direction cosines and not as the Euler or Cardan angles. The Euler or Cardan angles are computed only at the end of the iterations. The orientation of the platform can be computed once during one iterative step or can be re-calculated after computation of each point – this means three times per one iterative step. The latter one improves the iteration process.

The iteration progress is shown in Fig. 6 and 7. The convergence of the method is very robust – the number of the iteration steps depends on the initial iteration only imperceptibly.

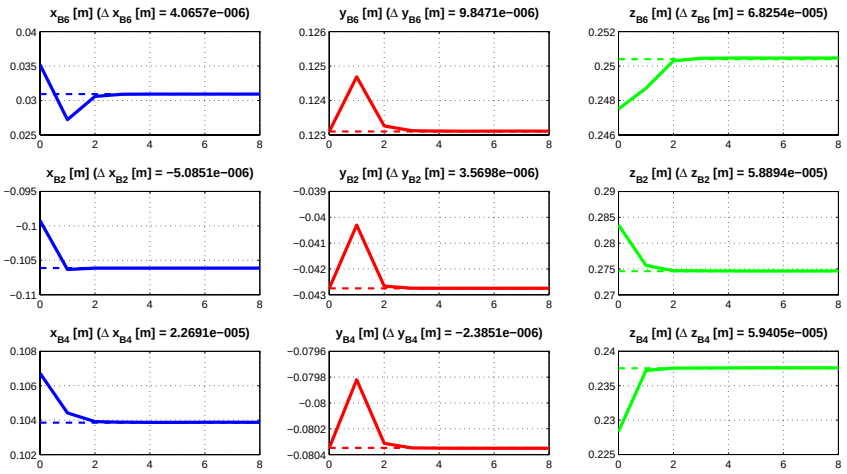

Figure 6 Iteration process with fixed initial iteration

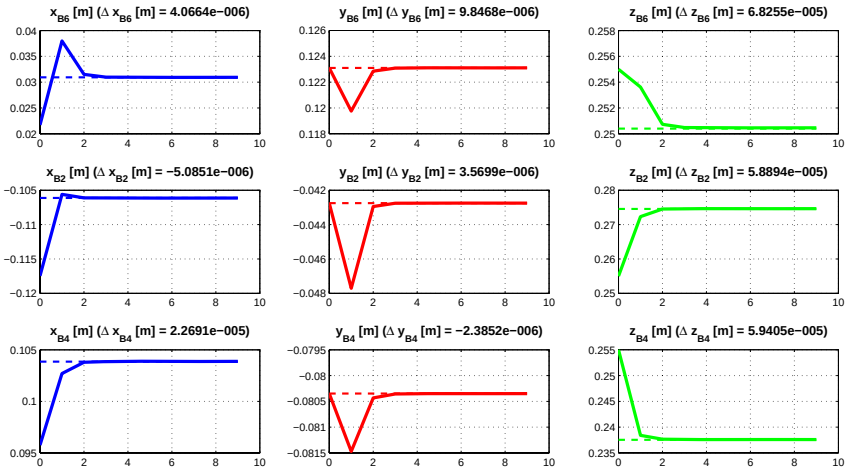

Figure 7 Iteration process with initial iteration equals previous step on trajectory

This procedure has been compared with traditional solution by Newton iterations. The comparison of the computational complexity (elapsed time) is in Tab. 1. The method of structural approximation is favourable and robust, especially for solving without any estimation. The reason for that is that no computation of Jacobi matrix and no Gauss elimination are necessary.

Table 1 Comparison of computational complexity

	Fixed initial iteration	Estimated initial iteration
Newton's iterative method	20.4 ms	5.5 ms
Structural approximation	9.2 ms	8.7 ms

The biggest issue of redundant mechanism forward kinematics is that the exact solution generally does not exist. As was mentioned above it is necessary to solve an overconstrained system of equations. The exact solution exists only for special sets of inputs (actuator positions s_i). However, the structural approximation seems to be very efficient technique to solve such a problem. Let us consider each of the actuator position s_i is with 1% error. The advantage of the structural approximation is that the computational complexity is influences very slightly (Tab. 2).

Table 2 Comparison of computational complexity for solution with errors of inputs

	Fixed initial iteration	Estimated initial iteration
Newton's iterative method	19.2 ms	6.0 ms
Structural approximation	9.5 ms	8.0 ms

6. Conclusion

The paper describes a new procedure of structural approximation for the solution of positional kinematical solution of parallel kinematical structures that are not analytically solvable. This procedure achieves better computational complexity than the traditional Newton method. The procedure has been used for the solution of forward kinematical problems of the HexaSphere.

7. Acknowledgement

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