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THE INFLUENCE OF THE CHOICE OF A CONSTITUTIVE RELATION IN THE MODELLING OF THE WAVE PROPAGATION IN PRE-STRESSED MEDIUM

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Summary: *The most common constitutive relation in acoustoelasticity is the second-order constitutive relation expressed in terms of the Green-Lagrange strain tensor. The wave velocities in isotropic elastic media for the case of homogeneous pre-stress were derived by Hughes and Kelly (1953). This material model is characterized by the high sensitivity of material parameters to the small measurement errors. This rather bad property leads to the proposal of another material model with the second-order constitutive relation expressed in terms of the logarithmic strain tensor. For this material model the acoustic wave velocities were derived for three different types of homogeneous pre-stress.*

1. Introduction

Acoustoelasticity is a phenomenon describing the pre-stress dependency of the velocities of sound-like waves propagating through an elastic material. The theory of acoustoelasticity was introduced by (Hughes and Kelly, 1953). They measured the effect of the uniaxial stress on the velocity of the wave in an isotropic elastic material. They also proposed the method for identification of the three additional third-order elastic constants corresponding to Murnaghan's free energy function (Murnaghan, 1951).

The frequently used Green-Lagrange strain tensor is easy and straightforward in its definition and may well be used, the constitutive models based on it often exhibit definite instabilities when performing large deformation.

Theoretically, all the strain measures are equivalent but for a fixed choice of the stored energy function, for instance a polynomial of the third-order, different strain tensors will represent different material models. If so, the definition of the strain tensor becomes essential and, in particular, stability then seems to be the key issue. Because of these reasons and keeping stability issues in mind, the authors of this paper set out to derive acoustic tensors based on the third order polynomial free energy function, using an arbitrary Hill strain measure, seeking the best alternative in terms of sensitivity property.

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In this work, the coefficients of the acoustic tensor as functions of the applied initial stress are determined. The result may be used to establish numerical values of material constants suitable for the logarithmic model of hyperelasticity.

2. Wave velocities in elastic media

They are two different kinds of waves propagating in the unbounded elastic media. If the motion of particles is in the same direction as is the direction of the wave propagation, then the wave is called pure longitudinal, and if the wave motion is perpendicular to the direction of the wave propagation, then the wave is called pure shear wave. The velocity of these waves propagating in an isotropic or orthotropic material with the principal material directions 1, 2 and 3, is given by

$$\begin{aligned} \rho_0 c_{0\mathrm{L}}^2 &= A_{1111} \\ \rho_0 c_{0\mathrm{S1}}^2 &= A_{2121} \\ \rho_0 c_{0\mathrm{S2}}^2 &= A_{3131}, \end{aligned} \tag{1}$$

where ρ_0 is the material density in the unloaded state, $c_{0\mathrm{L}}^2$ is the square of the 'natural' velocity of the longitudinal wave, $c_{0\mathrm{S1}}^2$ and $c_{0\mathrm{S2}}^2$ are the 'natural' velocities of the shear waves. 'Natural' velocity is related to the undeformed configuration, which can be more readily obtained from the experiments. A_{1111}, A_{2121} and A_{3131} are the components of the first elasticity tensor $\mathbf{A}$, which is defined (Ogden, 1984) as

$$\boldsymbol{\Pi} = \mathbf{A}\dot{\mathbf{F}}, \tag{2}$$

where $\boldsymbol{\Pi}$ is the first Piola-Kirchhoff stress tensor and $\dot{\mathbf{F}}$ is the material derivative of the deformation gradient $\mathbf{F}$. The cartesian components of the first elasticity tensor $\mathbf{A}$ can be for a general strain tensor $\mathbf{E}$ found in (Kruisova and Plesek, 2006). The components used in (1) have the forms

$$\begin{aligned} A_{1111} &= \Sigma_{11} f''(\lambda_1) + \left(f'(\lambda_1)\right)^2 H_{1111}, \\ A_{2121} &= \frac{\Sigma_{11} f'(\lambda_1)\lambda_1 - \Sigma_{22} f'(\lambda_2)\lambda_2}{\lambda_1^2 - \lambda_2^2} - 2\lambda_2^2 \frac{f(\lambda_1) - f(\lambda_2)}{\left(\lambda_1^2 - \lambda_2^2\right)^2} \left(\Sigma_{11} - \Sigma_{22}\right) + \\ &\quad + \lambda_2^2 \frac{\left[f(\lambda_1) - f(\lambda_2)\right]^2}{\left(\lambda_1^2 - \lambda_2^2\right)^2} \left(H_{1212} + H_{2112} + H_{2121} + H_{1221}\right), \\ A_{3131} &= \frac{\Sigma_{11} f'(\lambda_1)\lambda_1 - \Sigma_{33} f'(\lambda_3)\lambda_3}{\lambda_1^2 - \lambda_3^2} - 2\lambda_3^2 \frac{f(\lambda_1) - f(\lambda_3)}{\left(\lambda_1^2 - \lambda_3^2\right)^2} \left(\Sigma_{11} - \Sigma_{33}\right) + \\ &\quad + \lambda_3^2 \frac{\left[f(\lambda_1) - f(\lambda_3)\right]^2}{\left(\lambda_1^2 - \lambda_3^2\right)^2} \left(H_{1313} + H_{3113} + H_{3131} + H_{1331}\right), \end{aligned} \tag{3}$$

where λ_i are the principal stretches of the deformation, $f(\lambda_i)$ are the principal values of the general Hill strain tensor $\mathbf{E}$, $f'(\lambda_i)$ are the derivative of $f(\lambda_i)$ along the principal stretch λ_i. Σ_{ij} are the components of the stress tensor conjugate to the general Hill strain tensor, such as $JE_{ij}\Sigma_{ij} = \sigma_{kl}D_{kl}$ (J is the Jacobian of the deformation $J = \det \mathbf{F}$, σ_{kl} are the components of the Cauchy stress tensor and D_{kl} are the components of rate of the deformation tensor). The fourth order tensor $\mathbf{H}$ is the Hessian of the deformation energy ψ in respect to $\mathbf{E}$, its components are

$$H_{ijkl} = \frac{\partial^2 \psi}{\partial E_{ij} \partial E_{kl}}. \tag{4}$$

Since the general Hill's strain tensor is symmetric, the Hessian possesses both the major and minor symmetries.

3. Constitutive relations

The constitutive relations used here were of the second-order given by a third-order strain energy function ψ. Hereafter we restrict our investigation to the isotropic material. Then the strain energy function can be expressed in terms of three invariants

$$I_k = \frac{1}{k}\,\mathrm{tr}\,\mathbf{E}^k, \qquad k = 1,2,3 \tag{5}$$

of an arbitrary strain tensor $\mathbf{E}$ in the form

$$\psi = \frac{1}{2}\Lambda I_1^2 + 2\mu I_2 + \frac{1}{6}(2l - 2m + n)I_1^3 + (2m - n)I_1 I_2 + nI_3, \tag{6}$$

where Λ and μ and two Lamé's constants and l, m and n are the third order parameters. The stress tensor

$$\Sigma_{ij} = \frac{\partial \psi}{\partial E_{ij}} \tag{7}$$

is then

$$\Sigma_{ij} = \Lambda I_1 \delta_{ij} + \frac{1}{2}(2l - 2m + n)I_1^2 \delta_{ij} + (2m - n)I_2 \delta_{ij} + \\ + 2\mu E_{ij} + (2m - n)I_1 E_{ij} + nE_{im}E_{mj}. \tag{8}$$

The Hessian components are according to (4)

$$\begin{aligned} H_{1111} &= \Lambda + 2\mu + 2lI_1 + 4mE_{11}, \\ H_{1212} &= \frac{\mu}{2} + \frac{m}{2}I_1 - \frac{n}{4}E_{33}, \\ H_{1313} &= \frac{\mu}{2} + \frac{m}{2}I_1 - \frac{n}{4}E_{33}. \end{aligned} \tag{9}$$

4. Green-Lagrange strain tensor

If the Green-Lagrange strain tensor is used in conjunction with the second order constitutive relations, we talk about the Murnaghan material model and its new material parameters l, m, n are called Murnaghans parameters, see (Murnaghan, 1951). Since the material model of the second order with the Green-Lagrange strain tensor counts among the classic models in acoustoelasticity, the results are known, see (Hughes and Kelly, 1953). In this paper, the wave velocities were derived for the case of homogeneously pre-stressed material. The linearised relations of the wave velocities are given by

- hydrostatic pressure p

$$\rho_0 (c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu - \frac{p}{3\Lambda + 2\mu}(5\Lambda + 6\mu + 6l + 4m)$$

$$\rho_0 (c_{0\,\mathrm{S}})^2 = \mu - \frac{p}{3\Lambda + 2\mu}\left(3\Lambda + 4\mu + 3m - \frac{1}{2}n\right),$$

- pre-stress t in longitudinal direction

$$\rho_0(c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu + \left[9\Lambda + 6\mu + 4m + 2l + \frac{2\Lambda^2}{\mu} + \frac{4m\Lambda}{\mu}\right]\frac{t}{3\Lambda + 2\mu}$$
$$\rho_0(c_{0\,\mathrm{S}})^2 = \mu + \left[2\Lambda + 2\mu + m + \frac{n\Lambda}{4\mu}\right]\frac{t}{3\Lambda + 2\mu},$$

- pre-stress t in transverse direction

$$\rho_0(c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu + \left[2l - 2\Lambda - (\Lambda + 2m)\frac{\Lambda}{\mu}\right]\frac{t}{3\Lambda + 2\mu}$$
$$\rho_0(c_{0\,\mathrm{S1}})^2 = \mu + \left[m - \frac{n}{2} - \Lambda - \frac{n\Lambda}{2\mu}\right]\frac{t}{3\Lambda + 2\mu}$$
$$\rho_0(c_{0\,\mathrm{S2}})^2 = \mu + \left[\frac{n\Lambda}{4\mu} + 2\Lambda + 2\mu + m\right]\frac{t}{3\Lambda + 2\mu}.$$

5. Logarithmic strain tensor

Now the logarithmic strain tensor $\ln \mathbf{U}$ is used in the constitutive relations instead of the Green-Lagrange strain tensor and the wave velocities for homogeneous deformation are derived.

We suppose the homogeneous deformation described by the diagonal deformation gradient $\mathbf{F} = \mathrm{diag}\,[\lambda_1\ \ \lambda_2\ \ \lambda_3]$. The deformation gradients and the Cauchy stress tensors describing the loading modes analysed are

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \tag{10}$$

for the pre-stressed caused by the hydrostatic pressure,

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \qquad \boldsymbol{\sigma} = \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

for the material pre-stressed in the direction 1, which is identical with the direction of the wave propagation and

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \qquad \boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

for the material pre-stressed in the direction 2, which is the direction perpendicular to the direction of the wave propagation.

For the deformation gradient in the diagonal for the general constitutive relations (8) yields the components of the stress tensor $\mathbf{T}$, which is conjugate to the logarithmic strain tensor in the form

$$T_{11} = \ln\lambda_1\left(\Lambda + 2\mu\right) + 2\ln\lambda_2\Lambda + \ln^2\lambda_1\left(\bar{l} + 2\bar{m}\right) + \ln^2\lambda_2\left(4\bar{l} - 2\bar{m} + \bar{n}\right) + \ln\lambda_1\ln\lambda_2 4\bar{l} \tag{13}$$

and

$$T_{22} = T_{33} = \ln\lambda_1\Lambda + 2\ln\lambda_2\left(\Lambda+\mu\right) + \ln^2\lambda_1 l + \ln^2\lambda_2\left(4\bar{l}+2\bar{m}\right) + \ln\lambda_1\ln\lambda_2\left(4\bar{l}-2\bar{m}+\bar{n}\right). \tag{14}$$

Since the elastic material is supposed to be isotropic, the components of the stress tensor conjugate to the logarithmic strain tensor can be calculated from the Cauchy stress tensor through

$$T_{ij} = J R^{-1}_{ik}\sigma_{kl}R^{-1}_{jl}, \tag{15}$$

where $\mathbf{R} = \mathbf{I}$ for the strain described by the diagonal deformation gradient.

The use of Eqs. (10), (11), (12), (13), (14), and (15) yields the relations between the load and stretches. For hydrostatic pressure we have load versus stretch relation in form

$$-p = \lambda^{-3}\left[(3\Lambda+2\mu)\ln\lambda + \left(9\bar{l}+\bar{n}\right)\ln^2\lambda\right]. \tag{16}$$

The material pre-stressed in the longitudinal direction is described by the one-dimensional load versus stretch relation

$$\lambda_1\lambda_2^2\, t = \ln\lambda_1\left(\Lambda+2\mu\right) + 2\ln\lambda_2\Lambda + \ln^2\lambda_1\left(\bar{l}+2\bar{m}\right) + \ln^2\lambda_2\left(4\bar{l}-2\bar{m}+\bar{n}\right) + \ln\lambda_1\ln\lambda_2 4\bar{l} \tag{17}$$

and transverse versus longitudinal stretch relation

$$0 = \ln\lambda_1\Lambda + 2\ln\lambda_2\left(\Lambda+\mu\right) + \ln^2\lambda_1\bar{l} + \ln^2\lambda_2\left(4\bar{l}+2\bar{m}\right) + \ln\lambda_1\ln\lambda_2\left(4\bar{l}-2\bar{m}+\bar{n}\right). \tag{18}$$

The material pre-stressed in the transverse direction is described by the one-dimensional load versus stretch relation

$$\lambda_1^2\lambda_2\, t = 2\ln\lambda_1\Lambda + \ln\lambda_2\left(\Lambda+2\mu\right) + \ln^2\lambda_1\left(4\bar{l}-2\bar{m}+\bar{n}\right) + \ln^2\lambda_2\left(\bar{l}+2\bar{m}\right) + \ln\lambda_1\ln\lambda_2 4\bar{l} \tag{19}$$

and transverse versus longitudinal stretch relation

$$0 = 2\ln\lambda_1\left(\Lambda+\mu\right) + \ln\lambda_2\Lambda + \ln^2\lambda_1\left(4\bar{l}+2\bar{m}\right) + \ln^2\lambda_2\bar{l} + \ln\lambda_1\ln\lambda_2\left(4\bar{l}-2\bar{m}+\bar{n}\right). \tag{20}$$

For the hydrostatic pre-stress, the components of the first moduli given in (3) defining the wave velocities (1) are

$$\begin{aligned} \rho_0(c_{0\,\mathrm{L}})^2 &= \frac{1}{\lambda^2}\left[\Lambda+2\mu+\left(-3\Lambda-2\mu+6\bar{l}+4\bar{m}\right)\ln\lambda - \left(9\bar{l}+\bar{n}\right)\ln^2\lambda\right] \\ \rho_0(c_{0\,\mathrm{S}})^2 &= \frac{1}{\lambda^2}\left[\mu+\left(3\bar{m}-\frac{\bar{n}}{2}\right)\ln\lambda\right]. \end{aligned} \tag{21}$$

For the longitudinal pre-stress

$$\begin{aligned} \rho_0(c_{0\,\mathrm{L}})^2 &= -\lambda_1^{-1}\lambda_2^2\, t + \lambda_1^{-2}\left(\Lambda+2\mu\right) + \lambda_1^{-2}\ln\lambda_1\left(2\bar{l}+4\bar{m}\right) + \lambda_1^{-2}\ln\lambda_2 4\bar{l} \\ \rho_0(c_{0\,\mathrm{S}})^2 &= \frac{\lambda_1\lambda_2^2\, t}{\lambda_1^2-\lambda_2^2} - \frac{2\lambda_1\lambda_2^4\, t\left(\ln\lambda_1-\ln\lambda_2\right)}{\left(\lambda_1^2-\lambda_2^2\right)^2} + 4\mu\lambda_2^2\frac{\left(\ln\lambda_1-\ln\lambda_2\right)^2}{\left(\lambda_1^2-\lambda_2^2\right)^2} + \\ &\quad + 4\bar{m}\lambda_2^2\ln\lambda_1\frac{\left(\ln\lambda_1-\ln\lambda_2\right)^2}{\left(\lambda_1^2-\lambda_2^2\right)^2} + \left(8\bar{m}-2\bar{n}\right)\lambda_2^2\ln\lambda_2\frac{\left(\ln\lambda_1-\ln\lambda_2\right)^2}{\left(\lambda_1^2-\lambda_2^2\right)^2} \end{aligned} \tag{22}$$

and for the material pre-stressed in transversal direction

$$
\begin{aligned}
\rho_0(c_{0\,\mathrm{L}})^2 &= \lambda_1^{-2}\left[\Lambda + 2\mu + \ln\lambda_1\left(4\bar{l} + 4\bar{m}\right) + \ln\lambda_2\, 2\bar{l}\right]\\
\rho_0(c_{0\,\mathrm{S1}})^2 &= \lambda_1^{-2}\left[\mu + \ln\lambda_1\, 2\bar{m} + \ln\lambda_2\,\frac{2\bar{m} - \bar{n}}{2}\right]\\
\rho_0(c_{0\,\mathrm{S2}})^2 &= \frac{-\lambda_1^2\lambda_2\, t}{\lambda_1^2 - \lambda_2^2}\left[1 - 2\lambda_2^2\frac{\ln\lambda_1 - \ln\lambda_2}{\lambda_1^2 - \lambda_2^2}\right] + \\
&\qquad + \lambda_2^2\frac{\left[\ln\lambda_1 - \ln\lambda_2\right]^2}{\left[\lambda_1^2 - \lambda_2^2\right]^2}\left[4\mu + \ln\lambda_1\left(8\bar{m} - 2\bar{n}\right) + \ln\lambda_2\left(4\bar{m}\right)\right].
\end{aligned}
\tag{23}
$$

The relations (21), (22) and (23) are very complicated and can be used only together with non-linear relations (16) to (20), but analogously as in (Hughes and Kelly, 1953) they can be linearised for the negligible load causing the pre-stress $t \rightarrow 0$, $p \rightarrow 0$ and the stretch $\lambda \rightarrow 1$, $\lambda_1 \rightarrow 1$, and $\lambda_2 \rightarrow 1$.

The linearisation of the wave velocities for the elastic isotropic material pre-stressed in the transverse direction is shown in Fig. 1–3.

6. Results

The linearised velocities of the waves in case when the logarithmic strain is used instead of the Green-Lagrange strain are

- hydrostatic pressure p

$$
\begin{aligned}
\rho_0(c_{0\,\mathrm{L}})^2 &= \Lambda + 2\mu - \frac{p}{3\Lambda + 2\mu}\left(-5\Lambda - 6\mu + 6\bar{l} + 4\bar{m}\right)\\
\rho_0(c_{0\,\mathrm{S}})^2 &= \mu - \frac{p}{3\Lambda + 2\mu}\left(-2\mu + 3\bar{m} - \frac{\bar{n}}{2}\right),
\end{aligned}
$$

- pre-stress t in longitudinal direction

$$
\begin{aligned}
\rho_0(c_{0\,\mathrm{L}})^2 &= \Lambda + 2\mu + \left[-9\Lambda - 6\mu + 4\bar{m} + 2\bar{l} + \left(-2\Lambda + 4\bar{m}\right)\frac{\Lambda}{\mu}\right]\frac{t}{3\Lambda + 2\mu}\\
\rho_0(c_{0\,\mathrm{S}})^2 &= \mu + \left[-\frac{\Lambda}{2} - \mu + \bar{m} + \frac{\bar{n}\Lambda}{4\mu}\right]\frac{t}{3\Lambda + 2\mu},
\end{aligned}
$$

- pre-stress t in transverse direction

$$
\begin{aligned}
\rho_0(c_{0\,\mathrm{L}})^2 &= \Lambda + 2\mu + \left[2\Lambda + 2\bar{l} + \left(\Lambda - 2\bar{m}\right)\frac{\Lambda}{\mu}\right]\frac{t}{3\Lambda + 2\mu}\\
\rho_0(c_{0\,\mathrm{S1}})^2 &= \mu + \left[\Lambda + \bar{m} - \frac{\bar{n}}{2} - \frac{\bar{n}\Lambda}{2\mu}\right]\frac{t}{3\Lambda + 2\mu}\\
\rho_0(c_{0\,\mathrm{S2}})^2 &= \mu + \left[-\frac{\Lambda}{2} - \mu + \bar{m} + \frac{\bar{n}\Lambda}{4\mu}\right]\frac{t}{3\Lambda + 2\mu}.
\end{aligned}
$$

7. Acknowledgment

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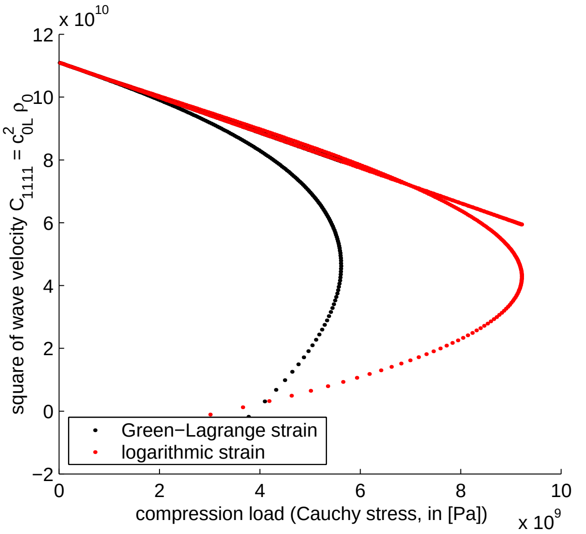

Figure 1: The longitudinal wave in transversally pre-stressed material.

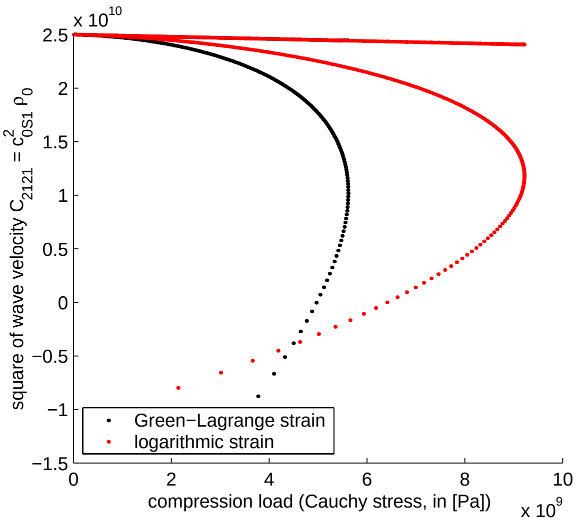

Figure 2: First shear wave in transversally pre-stressed material.

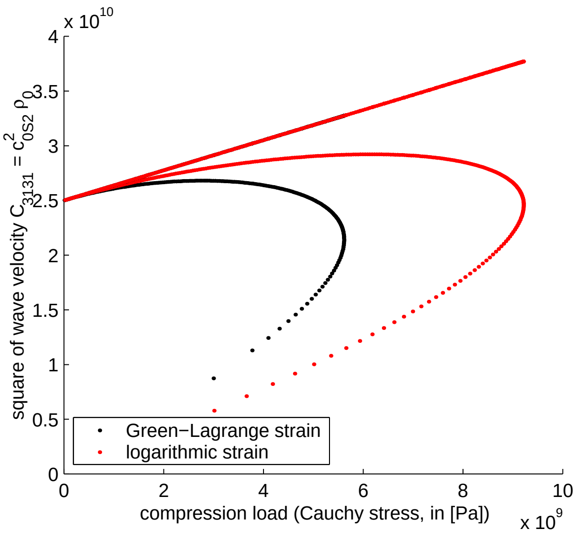

Figure 3: Second shear wave in transversally pre-stressed material.

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