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DETERMINATION OF FLOW RATE OF VISCOPLASTIC FLUIDS IN ANNULI USING AN ADDITIVE METHOD

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Summary: *An additive method enabling a direct determination of axial flow rate for viscous (power-law model) and viscoplastic (Vočadlo model) fluids in concentric annuli is presented. Flow rate through the actual concentric annulus is possible to express as a sum of axial flow rates through individual partial concentric annuli forming the whole annulus. The resulting relation for axial flow rate is possible to express in analytical forms without necessity of otherwise complicated derivations. The only numerical calculation consists in a determination of location of a zero shear stress for the original whole concentric annulus.*

1. Introduction

Calculations of fully developed, laminar shear flows of non-Newtonian fluids through channels such as circular pipes and annuli are commonly needed both in the determination of flow conditions and deformation response and, consequently, in the design of equipment handling these fluids. Thorough analysis of these flow situations also serves as an important introductory reference step for studies of more complicated flows.

Flow in channels mentioned above is often encountered in various industrial processes such as transportation of drilling fluids in petroleum industry or extrusion.

In all these cases fluids used very often exhibit rheological behavior of visco-elasto-plastic type. Therefore, we restrict in the following to fluids that can be regarded (or best approximated) as visco-elasto-plastic. In the past various models were proven to express rheological properties of these fluids, each model characterizing more efficiently a special individual family of fluids under consideration - as e.g. polymer melts and solutions, filled polymers, elastomers, pastes, oils, drilling fluids, etc.

In our contribution we concentrate on flow through annular passages challenging a series of hitherto unsolved problems even for isothermal, incompressible fluids. Annular flow is characterized by the inhomogeneous distribution of the shear stresses in the annular region in contrast to the homogeneous distribution in plate slot flow. Shear stresses are generated by

- imposing pressure forces (Poiseuille flow);
- applying drag forces (Couette flow);
- combination of the pressure and drag forces (Couette-Poiseuille flow).

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Each of the two forces is generally composed of two components - axial and tangential - depending on the action of the respective force. As no superposition principle takes place in Couette-Poiseuille annular flows, it is necessary to solve separately each combination of both Couette and Poiseuille flow.

In addition, a difficulty in any flow calculations with viscoplastic fluids arises in the determination of possible plug zones in which no flow deformation occurs. For example, in axial annular flow with no boundary movement there is always a detached moving plug of unyielded fluid, provided there is any flow at all.

Couette flow is usually realized either by a torque causing the rotation of the inner or outer cylinder, or by a drag force applied in actual direction to the individual cylinders. Poiseuille flow is generated by the pumping devices supplying the fluid. In practice, usefulness of the solution of the individual flow arrangements is measured by ´quality´ of the relation between volumetric flow rate and driving forces in which rheological, geometrical, and kinematical parameters are involved. For some arrangements (e.g. axial pressure flow, axial pressure flow with axially moving inner cylinder) using simpler rheological models a few authors succeeded in deriving an explicit analytical relation flow rate vs. driving forces; an overwhelming majority of the authors obtained relations in the form of definite integrals necessitating numerical quadrature, or they used various computational methods (as e.g. FEM). However, the quality of such solutions (otherwise very ingenious) suffers from two shortcomings. Firstly, with arbitrary small change of any parameter it is necessary to repeat the whole computational procedure with respect to strong nonlinearity of the given problem, and thus the non-predicability of the measure of the solution change. Secondly, unlike the explicit analytical relations, these numerical solutions lack of direct insight into the nature of the problem; especially the measure of influence of individual entry parameters is not so obvious.

The complexity of problems is further intensified for the case of eccentrically arranged cylinders.

The advantage of analytical approaches (if they are possible to apply) over the numerical methods is obvious also for the case of viscoplastic fluids. These types of flow situations exhibit the so-called plug regions where no flow deformation occurs. The arrangements of these plug flows can be determined by means of criteria derived analytically from the stress distribution prior to the detailed analysis of the velocity field.

Summarizing these paragraphs there is still a lot of problems yet unsolved and challenging the need of continuation of basic research in this area.

In practice these types of problems are encountered e.g. in wellbore drilling. With respect to demand of continuous uniform flow rate it is necessary to know flow rates through arbitrary annular cross sections of the whole annulus. The hint how to calculate these flow rates for fluid obeying the Vočadlo model (for which a power-law model forms its subcase) is given in the following section.

We suppose that flow is steady, laminar, incompressible, isothermal and axial with negligible end effects of the inner and outer cylinders.

2. Problem formulation

Under the above stated assumptions the balance equation is of the form

$$\frac{1}{r}\frac{d(r\tau_{rz})}{dr}=P \tag{1}$$

with the boundary conditions

$$v_z(\kappa R)=V>0, \quad v_z(R)=0 \tag{2}$$

and the Vočadlo model [Parzonka & Vočadlo, 1967] (written in the form useful for the cylindrical coordinates)

$$\tau_{rz}=\left[K^{\frac{1}{n}}\left|\frac{dv_z}{dr}\right|^{\frac{n-1}{n}}+\tau_0^{\frac{1}{n}}\left|\frac{dv_z}{dr}\right|^{-\frac{1}{n}}\right]^n\left(-\frac{dv_z}{dr}\right) \qquad \text{for} \quad |\tau_{rz}|\geq\tau_0 \quad ,$$

$$\frac{dv_z}{dr}=0 \qquad \text{for} \quad |\tau_{rz}|\leq\tau_0 \quad , \tag{3}$$

where P represents pressure gradient in an axial direction, R (κR) is a radius of the outer (inner) cylinder, τ_0 is a yield stress.

As mentioned in the Introduction the problem (1)-(3) was solved for power-law fluid (τ_0=0) by Malik & Shenoy (1991), and for $\tau_0\neq0$ by David & Filip (2003). In both cases – in spite of analytical forms for determination of flow rates q – there is a necessity to calculate a parameter λ (at location λR shear stress nullifies) from the corresponding integral equation. Knowledge of this parameter enables to determine velocity fields across the whole annulus (see Malik & Shenoy (1991), David & Filip (2003)).

Using the following transformations

$$\xi=\frac{r}{R}, \quad \phi=\frac{v_z}{V}, \quad T=\frac{2\tau_{rz}}{|P|R}, \quad T_0=\frac{2\tau_0}{|P|R}, \quad \Lambda=\frac{|P|R}{2K}\left(\frac{R}{V}\right)^n, \quad Q=\frac{q}{2\pi R^2 V} \tag{4}$$

the problem (1)-(3) can be converted to the dimensionless form

$$(\mathrm{sgn}(P))\cdot T=\xi-\frac{\lambda^2}{\xi} \quad , \tag{5}$$

$$\phi(\kappa)=1, \quad \phi(1)=0 \quad , \tag{6}$$

$$T=\left[\Lambda^{-s}\left|\frac{d\phi}{d\xi}\right|^{1-s}+T_0^{-s}\left|\frac{d\phi}{d\xi}\right|^{-s}\right]^n\left(-\frac{d\phi}{d\xi}\right) \qquad \text{for} \quad |T|\geq T_0 \quad , \tag{7}$$

$$\frac{d\phi}{d\xi}=0 \qquad \text{for} \quad |T|\leq T_0 \tag{8}$$

where λ^2 is a dimensionless constant of integration.

If λ_i, λ_o denote the boundary values of the plug flow region, then from rel.(5) it follows that

$$\lambda^2 = \lambda_i \lambda_o \quad , \tag{9}$$

$$\lambda_i = \lambda_o - T_0 \quad . \tag{10}$$

From here we obtain - for the case when imposed pressure gradient assists the drag of the inner cylinder – the velocity gradient in the form

$$\frac{d\phi_i}{d\xi} = \Lambda^s \left[\left(\frac{\lambda^2}{\xi} - \xi \right)^s - T_0^s \right] \quad \text{for} \quad \kappa \le \xi < \lambda_i \quad (\text{where } \frac{d\phi}{d\xi} > 0) \quad , \tag{11}$$

$$\frac{d\phi_p}{d\xi} = 0 \quad \text{for} \quad \lambda_i \le \xi \le \lambda_o \quad , \tag{12}$$

$$\frac{d\phi_i}{d\xi} = -\Lambda^s \left[\left(\xi - \frac{\lambda^2}{\xi} \right)^s - T_0^s \right] \quad \text{for} \quad \lambda_o < \xi \le 1 \quad (\text{where } \frac{d\phi}{d\xi} < 0) \quad . \tag{13}$$

and for the case when imposed pressure gradient opposes the drag of the inner cylinder

$$\frac{d\phi_i}{d\xi} = -\Lambda^s \left[\left(\frac{\lambda^2}{\xi} - \xi \right)^s - T_0^s \right] \quad \text{for} \quad \kappa \le \xi < \lambda_i \quad (\text{where } \frac{d\phi}{d\xi} < 0) \quad , \tag{14}$$

$$\frac{d\phi_p}{d\xi} = 0 \quad \text{for} \quad \lambda_i \le \xi \le \lambda_o \quad , \tag{15}$$

$$\frac{d\phi_i}{d\xi} = \Lambda^s \left[\left(\xi - \frac{\lambda^2}{\xi} \right)^s - T_0^s \right] \quad \text{for} \quad \lambda_o < \xi \le 1 \quad (\text{where } \frac{d\phi}{d\xi} > 0) \quad . \tag{16}$$

If we want to calculate the volumetric flow rate q_1 for an arbitrary annular cross section ($\kappa_1 R_1, R_1$) of the original annulus ($\kappa R \le \kappa_1 R_1 < R_1 \le R$) – for notation see Fig.1 - it is necessary to realise the following

- balance equation is the same for both annuli (arbitrary and original)
- location of zero shear stress in dimensional form is the same, i.e. $\lambda R = \lambda_1 R_1 \rightarrow \lambda_1 = \lambda R / R_1$
- boundary conditions at the inner and outer cylinders of an arbitrary annulus are possible to obtain from the relations (11)-(13) or (14)-(16)
- value q_1 of flow rate for an arbitrary annulus is possible to calculate analytically using relations in Malik & Shenoy (1991) and David & Filip (2003).

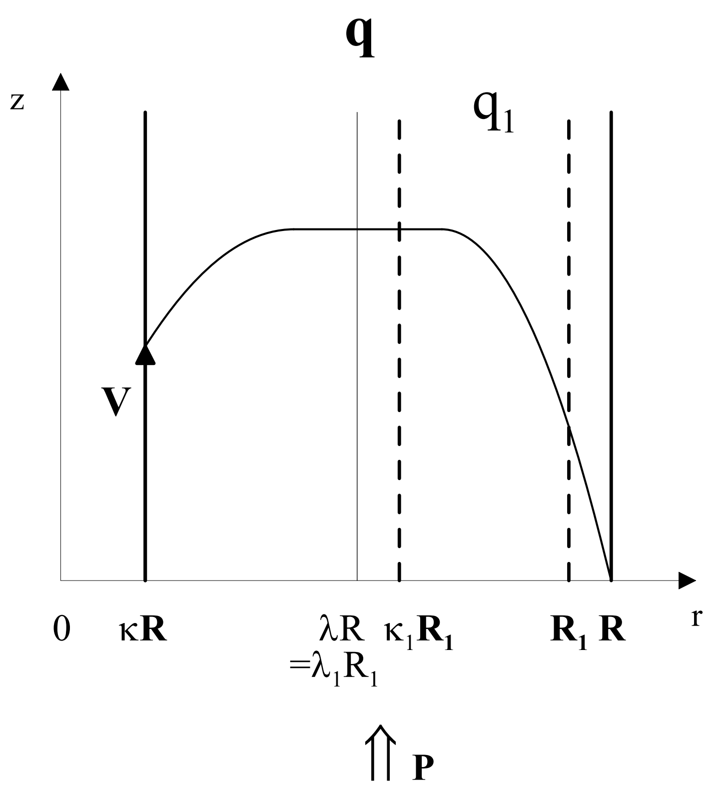

Figure 1 Definition sketch

3. Conclusion

The above section shows a procedure how it is possible to determine fully analytically the volumetric flow rate through an arbitrary annulus that forms the inner part of the original annulus, if we know for this original annulus a location of zero shear stress or a value of an integration constant λ^2. We suppose that fluid is of power-law or Vočadlo types.

4. Acknowledgment

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5. References

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