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VIBRATION OF THIN RECTANGULAR VISCOELASTIC ORTHOTROPIC PLATES UNDER TRANSVERSE NON-STATIONARY LOADING

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Summary: *The contribution is a part of the system investigation specialized to bending vibration of the rectangular thin 2D plate in case of analysis of the various model of plate influence, reological properties of the excitation loading and other initial assumptions. The solution of the plate for Kirchhoff, Rayleigh and Zener model of the standard body for general excitation loading. The approximate analytic solution for 2D will be compared with FEM solution of 3D plate for loading in form of Heavisid's function (jump).*

1. Introduction

The presented article is a part of systematic investigation of transient stress and deformation of the bodies made from nonconventional materials, first of all from plastic – polymers, composites etc. The amount of the utilization of these mentioned materials is continuously rising in the industry production. According to the specific properties of these materials, first of all the capability to dumping amplitudes of distributing excitements, the description demand and transient stress state and deformation in bodies modeling is growing. The work is aimed to the bodies with significant viscoelastic and anisotropic properties.

Already from first half of the twentieth century the attention is devoted to the theory progress of viscoelasticity are stated by Kolski, (1958) and wave phenomena investigation in two dimensional bodies.

In the second half of the twentieth century the investigation is extended with transient phenomena in two dimensional viscoelastic bodies are stated by Weaver, Sachse and Niu (1989) and in consequence to development of the elasticity theory of anisotropic bodies are stated by Lechnickij (1977), Hearmon (1965), Tiang (1996), Mamrilla and Mamrillova (1988), dynamic of the elastic anisotropic plates are stated by Hermon (1965) Lechnickij (1947, 1957), Ambarcumjan (1987) and statics of laminate anisotropic plates (Whitney, 1987) the solution is developed. At first the stationary later the transient solutions of the stress state and deformation of viscoelastic anisotropic (orthotropic at first of all) plates were determined are stated by Sobotka (1984), Volek 1990).

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The presented models which defining the solution assumptions enable creation of the system of particular various cases combination. The solutions and comparison of this cases make possible to analyze its influence to time and dimension field of searched quantities, above all the displacements velocities, stress components and deformation. The basic variation of all cases is model of elastic isotropic body from point of view based on reological properties of continuum.

The analytical approximate solution of elastic and viscoelastic plate for particular cases is compared with FEM solution for 3D plate which will performed in finite element method system MSE Marc (ÚT AV ČR Plzeň) and in selected cases will be both these solutions compared with results evaluated with experimental methods. The investigation will be performed by laser interferometer method or electric resistance tensiometers method (ÚT AV ČR Praha).

The significance of the input models parameters and assumption should be obtained from the systematic investigation and comparison of particular cases.

The approximate solution of the thin 2D plate loaded with jump force $F(t) = F_0.H(t)$ was performed fig.1 for combination of the plate models and material models.

A) Model of plate : Kirchhoff, Rayleighy
Aa) Model of material: Voigt-Kelvin, special orthotropy are stated by Soukup, Volek (2007a)
Ab) Model of material: Maxwell, special orthotropy are stated by Soukup, Volek (2007b) and Soukup, Volek and Skočilas (2007)

B) Model desky: Flüge, Mindlin
Ba) Model of material: Maxwell, special orthotropy are stated by Soukup, Volek and Skočilas (2007).

2. Methods

In presented contribution is devoted the problem solution for model of the plate: Kirchhoff and Rayleigh and model of plate material: Zener – standard body with special ortotropy.

The investigation of the transversal vibration of the plates using the approximate methods of thin 2D plate theory results from the assumed conditions. These assumptions define input parameters relations: the model of the plate geometry and its deformations are stated by Babuška and Li (1992), the model of the reological material properties, the model of the boundary conditions – the plate suspension, the time and space distribution of the external excitation vertical loading etc.

The fundamental problem is the acceptability of these simplifying assumptions. The question is how these assumptions affect the solution procedure and mostly the solution aim to obtain the dependence of fundamental mechanical quantities (i.e. displacement, deflection angles, velocities, accelerations, forces, moments, etc.) on time. The solution is demanded at the arbitrary place of the plate for the specified support and for the generally specified transient external transversal excitation loading.

We assumed the followed presumptions based on the solution of the several cases:

- the model of the rectangular 2D plate with less thickness than surface dimensions

- the model of the plate deformation – Kirchhoff-Love, Rayleigh, Flügge it is solved in Babuška (1992), Timoshenko-Mindlin-Reissner models it is solved in Reissner (1945) obr. 2

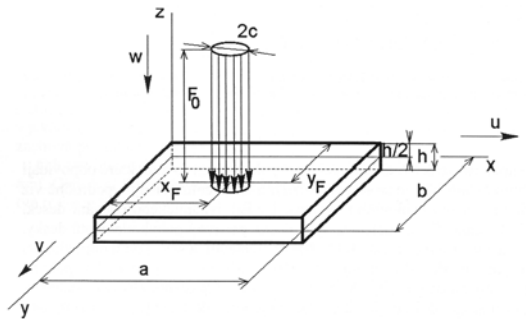


Fig. 1 Model under study

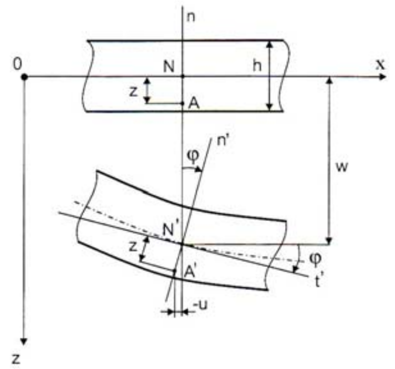
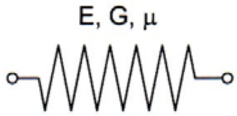


Fig. 2 Element of deflected plate

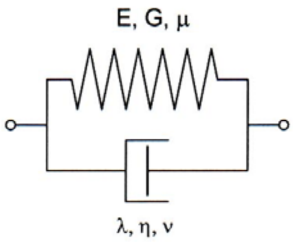
- the model of the simply supported plate – fig. 1
- the external exciting transversal transient loading defined by arbitrary integrable function F(t) – fig. 1.
- the model of the reological properties – the isotropic and anisotropic (special or general orthotropy) continuum, for linear models – elastic Hook, viscoelastic Voigt-Kelvin, Maxwell, Zener models, for generally anisotropic viscoelastic model, hereditary-materials – Volterra (it is solved in Volterra, 1951), Boltzmann, with limited possibility to obtain the parameters for particular models.

Hooke's model



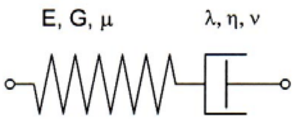
$$\sigma_i = b_{ij} \varepsilon_j$$

Voight-Kelvin's model



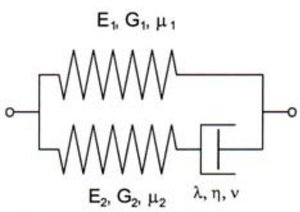
$$\sigma_i = b_{ij} \varepsilon_j + d_{ij} \frac{\partial \varepsilon_j}{\partial t}$$

Maxwell's model



$$\sigma_i + c_{ij} \frac{\partial \sigma_i}{\partial t} = d_{ij} \frac{\partial \varepsilon_j}{\partial t}$$

Zener's model



$$\sigma_i + c_{ij} \frac{\partial \sigma_i}{\partial t} = b_{ij} \varepsilon_j + d_{ij} \frac{\partial \varepsilon_j}{\partial t}$$

	Viscoelastic	Elastic
Anizotropic	$a_{ijkl} \sigma_{ij} + c_{ijkl} \dot{\sigma}_{ij} = b_{ijkl} \varepsilon_{kl} + d_{ijkl} \dot{\varepsilon}_{kl}$	$a_{ijkl} \sigma_{ij} = b_{ijkl} \varepsilon_{kl}$
Orthotropic	$\sigma_{ij} = b_{ij} \varepsilon_j + d_{ij} L_i(\varepsilon_j)$	$\sigma_i = b_{ij} \varepsilon_j$

Voight - Kelvin   Maxwell, Zener

$$L_i\left(\varepsilon_j(t)\right)=\frac{\partial \varepsilon_j}{\partial t} \qquad L_i\left(\varepsilon_j(t)\right)=\int_0^t K_i(t-\tau)\varepsilon_j(\tau)d\tau$$

Model of body material

Boltzmann

$$\sigma_{ij}=b_{ij}\varepsilon_j+d_{ij}L_i(\varepsilon_j) \qquad L_i\left(\varepsilon_j(t)\right)=\int_0^t K_i(t-\tau)\varepsilon_j(\tau)d\tau$$

Voltera

$$\sigma_{ij}=b_{ij}\varepsilon_j+d_{ij}L_i(\varepsilon_j) \qquad L_i\left(\varepsilon_j(t)\right)=\int_0^t K_i(t-\tau)\frac{\partial \varepsilon_j}{\partial \tau}d\tau$$

The equation for anisotropy linear viscoelastic material is stated in Shu and Onat (1965).

$$\sigma_i=d_{ij}L_i(\varepsilon_j) \qquad L_i\left(\varepsilon_j(t)\right)=\int_0^t K_i(t-\tau)\frac{\partial \varepsilon_j}{\partial \tau}d\tau$$

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix}=\begin{vmatrix}b_{11} & b_{12} & b_{13} & 0 & 0 & 0\\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0\\ b_{31} & b_{32} & b_{33v1} & 0 & 0 & 0\\ 0 & 0 & 0 & b_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & b_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & b_{66}\end{vmatrix}.\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix}+.$$

$$+\begin{vmatrix}d_{11}L_1 & d_{12}L_1 & d_{13}L_1 & 0 & 0 & 0\\ d_{21}L_2 & d_{22}L_2 & d_{23}L_3 & 0 & 0 & 0\\ d_{31}L_3 & d_{32}L_3 & d_{33}L_3 & 0 & 0 & 0\\ 0 & 0 & 0 & d_{44}L_4 & 0 & 0\\ 0 & 0 & 0 & 0 & d_{55}L_5 & 0\\ 0 & 0 & 0 & 0 & 0 & d_{66}L_6\end{vmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix} \tag{1}$$

$$\sigma_i=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}^T=\{\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz}\}^T \tag{2}$$

$$\varepsilon_j=\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6\}^T=\{\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{xz},\gamma_{yz}\}^T \tag{3}$$

Constitutive equation of the stress component in form of deformation components function

$$\sigma_x=b_{11}\varepsilon_x+b_{12}\varepsilon_y+d_{11}\int_0^t \varepsilon_x.e^{-\delta_x(t-\tau)}d\tau+d_{12}\int_0^t \varepsilon_x e^{-\delta_y(t-\tau)}d\tau$$

$$\sigma_y=b_{21}\varepsilon_x+b_{22}\varepsilon_y+d_{21}\int_0^t \varepsilon_x.e^{-\delta_y(t-\tau)}d\tau+d_{22}\int_0^t \varepsilon_y e^{-\delta_y(t-\tau)}d\tau \tag{4}$$

$$\tau_{xy}=b_{44}\gamma_{xy}+d_{44}\int_0^t \gamma_{xy}\ e^{-\delta_{xy}(t-\tau)}d\tau$$

where

$$b_{11} = b_{x1} + b_{x2}\,, \qquad b_{12} = b_{x1}\,\mu_{xy1} + b_{x2}\,\mu_{xy2}\,,$$

$$b_{21} = b_{y1}\,\mu_{xy1} + b_{y2}\,\mu_{xy2}\,, \qquad b_{22} = b_{y1} + b_{y2}\,,$$

$$b_{x1} = \frac{E_{x1}}{1-\mu_{yx1}\,\mu_{xy1}}\,, \qquad b_{y1} = \frac{E_{y1}}{1-\mu_{yx1}\,\mu_{xy1}}\,,$$

$$b_{x2} = \frac{E_{x2}}{1-\mu_{yx2}\,\mu_{xy2}}\,, \qquad b_{y2} = \frac{E_{y2}}{1-\mu_{yx2}\,\mu_{xy2}}\,,$$

$b_{ij} = G_1 + G_2,$ $\qquad i = j = 4,5,6$

$$d_{11}L_1 = \frac{-E_{x2}}{1-\nu_{yx}.\nu_{xy}}\delta_x \int_0^t \varepsilon_x\, e^{-\delta_x(t-\tau)}d\tau\,, \quad d_{12}L_1 = \frac{-E_{x2}.\nu_{xy}}{1-\nu_{yx}.\nu_{xy}}\delta_x \int_0^t \varepsilon_y\, e^{-\delta_x(t-\tau)}d\tau\,,$$

$$d_{21}L_2 = \frac{-E_{y2}.\nu_{yx}}{1-\nu_{yx}.\nu_{xy}}\delta_y \int_0^t \varepsilon_x\, e^{-\delta_y(t-\tau)}d\tau\,, \quad d_{22}L_2 = \frac{-E_{y2}}{1-\nu_{yx}.\nu_{xy}}\delta_y \int_0^t \varepsilon_y\, e^{-\delta_y(t-\tau)}d\tau$$

$$d_{ij}.L_i = -G_2\,\delta_{xy}\int_0^t \gamma_{xy}\, e^{-\delta_{xy}(t-\tau)}d\tau\,, \qquad i = j = 4,\,5,\,6$$

$$\delta_x = \frac{E_{x2}}{\lambda_x}\,, \qquad \delta_y = \frac{E_{y2}}{\lambda_y}\,, \qquad \delta_{xy} = \frac{G_2}{\eta}\,.$$

Stress components in form of displacement w function

$$\sigma_x = -z.\left(b_{11}\frac{\partial^2 w}{\partial x^2} + b_{12}\frac{\partial^2 w}{\partial y^2} + d_{11}\int_0^t \frac{\partial^2 w}{\partial x^2}e^{-\delta_x(t-\tau)}d\tau + d_{12}\int_0^t \frac{\partial^2 w}{\partial y^2}e^{-\delta_x(t-\tau)}d\tau\right)$$

$$\sigma_y = -z.\left(b_{21}\frac{\partial^2 w}{\partial x^2} + b_{22}\frac{\partial^2 w}{\partial y^2} + d_{21}\int_0^t \frac{\partial^2 w}{\partial x^2}e^{-\delta_y(t-\tau)}d\tau + d_{22}\int_0^t \frac{\partial^2 w}{\partial y^2}e^{-\delta_y(t-\tau)}d\tau\right) \tag{5}$$

$$\tau_{xy} = -2z.\left(b_{44}\frac{\partial^2 w}{\partial x.\partial y} + d_{44}\int_0^t \frac{\partial^2 w}{\partial x.\partial y}e^{-\delta_{xy}(t-\tau)}d\tau\right)$$

Momentums in form of displacement w function

$$m_x = -\left[D_x\frac{\partial^2 w}{\partial x^2} + D_{x\mu}\frac{\partial^2 w}{\partial y^2} - D_{x\nu}\delta_x\int_0^t \frac{\partial^2 w}{\partial x^2}e^{-\delta_x(t-\tau)}d\tau - D_{x\nu}\,\delta_x\,\nu_{xy}\int_0^t \frac{\partial^2 w}{\partial y^2}e^{-\delta_x(t-\tau)}d\tau\right]$$

$$m_y = -\left[D_{y\mu}\frac{\partial^2 w}{\partial x^2} + D_y\frac{\partial^2 w}{\partial y^2} - D_{y\nu}\delta_y\int_0^t \frac{\partial^2 w}{\partial x^2}e^{-\delta_y(t-\tau)}d\tau - D_{y\nu}\,\delta_y\,\nu_{yx}\int_0^t \frac{\partial^2 w}{\partial y^2}e^{-\delta_y(t-\tau)}d\tau\right] \tag{6}$$

$$m_{xy} = m_{yx} = -2\left[\left(D_{xy1} + D_{xy2}\right)\frac{\partial^2 w}{\partial x \partial y} - D_{xy2}\delta_{xy}\int_0^t \frac{\partial^2 w}{\partial x \partial y}e^{-\delta_{xy}(t-\tau)}d\tau\right]$$

where

$$D_x = \frac{h^3}{12}\left[\frac{E_{x1}}{1-\mu_{yx1}\,\mu_{xy1}} + \frac{E_{x2}}{1-\mu_{yx2}\mu_{xy2}}\right], \qquad D_{x\mu} = \frac{h^3}{12}\left[\frac{E_{x1}\,\mu_{xy1}}{1-\mu_{yx1}\,\mu_{xy1}} + \frac{E_{x2}\,\mu_{xy2}}{1-\mu_{yx2}\mu_{xy2}}\right],$$

$$D_{y\mu} = \frac{h^3}{12}\left[\frac{E_{y1}\,\mu_{yx1}}{1-\mu_{yx1}\,\mu_{xy1}} + \frac{E_{y2}\,\mu_{yx2}}{1-\mu_{yx2}\mu_{xy2}}\right], \qquad D_y = \frac{h^3}{12}\left[\frac{E_{y1}}{1-\mu_{yx1}\,\mu_{xy1}} + \frac{E_{y2}}{1-\mu_{yx2}\mu_{xy2}}\right],$$

$$D_{x\nu} = \frac{E_{x2}}{1-\nu_{yx}\nu_{xy}}\frac{h^3}{12}, \qquad D_{y\nu} = \frac{E_{y2}}{1-\nu_{yx}\nu_{xy}}\frac{h^3}{12},$$

$$\delta_x = \frac{E_x}{\lambda_x}, \qquad \delta_y = \frac{E_y}{\lambda_y}, \qquad \delta_{xy} = \frac{G_2}{\eta},$$

$$D_{xy1} = \frac{h^3}{12}G_1, \qquad D_{xy2} = \frac{h^3}{12}G_2, \qquad D_{xy} = D_{xy1} + D_{xy2}$$

The resulting motion equation of thin 2D plate

For Kirchhoff's model $\frac{\partial \Phi_x}{\partial x} = 0$, $\qquad \frac{\partial \Phi_y}{\partial y} = 0$

For Rayleigh's model $\frac{\partial \Phi_x}{\partial x} = \frac{\partial}{\partial x}\frac{\partial^2 w}{\partial t^2} \neq 0$, $\qquad \frac{\partial \Phi_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial^2 w}{\partial t^2} \neq 0$ (7)

In other form

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial x^2} + \frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y} - \rho h \frac{\partial^2 w}{\partial t^2} = -p(x;y;t) \tag{8}$$

after substituting to m_x, m_{xy}, m_y we arise to integral-differential equation

$$D_x \frac{\partial^4 w}{\partial x^4} + \left[D_{x\mu} + D_{y\mu} + 4D_{xy}\right]\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - D_x \delta_x \int_0^t \left(\frac{\partial^4 w}{\partial x^4} + \nu_{xy}\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) e^{-\delta_x (t-\tau)} d\tau -$$

$$- D_y \delta_y \int_0^t \left(\frac{\partial^4 w}{\partial y^4} + \nu_{yx}\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) e^{-\delta_y (t-\tau)} d\tau - 4D_{xy}\delta_{xy}\int_0^t \frac{\partial^4 w}{\partial x^2 \partial y^2} e^{-\delta_{xy}(t-\tau)} d\tau - \frac{\partial \Phi_x}{\partial x} - \frac{\partial \Phi_y}{\partial y} +$$

$$+ \rho h \frac{\partial^2 w}{\partial t^2} = -p(x;y;t) \tag{9}$$

For Kirchhoff's model $\qquad -\left(\frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y}\right) = 0$

For Rayleigh's model $$-\left(\frac{\partial \Phi_x}{\partial x}+\frac{\partial \Phi_y}{\partial y}\right)=-\rho\frac{h^3}{12}\frac{\partial^2}{\partial t^2}\nabla^2 w$$

Solution of equation (6) is possible to search by Fourrier's method in form double-consecution

$$w(x;y;t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W(t)X(x)Y(y) \tag{10}$$

Boundary condition of the simply supported rectangular plate satisfy by function

$$X(\alpha_m x)=\sin(\alpha_m x) \quad \text{where} \quad \alpha_m=\frac{m.\pi}{a}, \quad Y(\beta_n y)=\sin(\beta_n y) \quad \text{where} \quad \beta_n=\frac{n.\pi}{b}$$

Similarly it is possible to express the Fourrier's method the external excitation loading in form

$$p(x;y;t)=P(x;y)\,T_F(t)$$

where function $T_F(t)$ define time course of the excitation vertical loading and function $P(x;y)$ express its distribution on plate surface.

After substituting expression $w(x;y;t)$ according to (10) into the equation (9) and after carrying out the scalar product with regard to orthogonal function $X(\alpha_m x)$ and $Y(\beta_n y)$ we arise to integral-differential equation for function W(t) after modification.

$$\frac{d^2W(t)}{dt^2}+A_1W(t)-A_2\int_0^t W(\tau)e^{-\delta_x(t-\tau)}d\tau-A_3\int_0^t W(\tau)e^{-\delta_y(t-\tau)}d\tau-A_4\int_0^t W(\tau)e^{-\delta_{xy}(t-\tau)}d\tau=A_5T_F(t)$$

where

$$A_1=\left[D_x\alpha_m^4+\left(D_{x\mu}+D_{y\mu}+4D_{xy}\right)\alpha_m^2.\beta_n^2+D_y\beta_n^4\right]A^{-1}, \qquad A_2=D_x\delta_x\left(\alpha_m^4+\nu_{xy}\alpha_m^2\beta_n^2\right)A^{-1}$$

$$A_3=D_y\delta_y\left(\beta_n^4+\nu_{yx}\alpha_m^2\beta_n^2\right)A^{-1}, \qquad A_4=D_{xy}\delta_{xy}\alpha_m^2\beta_n^2\,A^{-1} \tag{11}$$

$$A_5=\frac{\int_0^a\int_0^b P(xy)\;X(\alpha_m x)\;Y(\beta_n y)\,dx.dy}{\int_0^a\int_0^b\left[X(\alpha\,x)\,Y(\beta_n y)\right]^2dx\,dy}A^{-1}$$

where $A=\rho h+0$ for Kirchhoff's model

$A=\rho h+\rho\frac{h^3}{12}\left(\alpha_m^2+\beta_n^2\right)$ for Rayleigh's model

After arrangement $\mathrm{A}=\rho\,h\,\Psi_{mn}$

where $\Psi_{mn}=1;$ for Kirchhoff's model

$\Psi_{mn}=1+\frac{h^2}{12}\left(\alpha_m^2+\beta_n^2\right)$ for Rayleigh's model

For supposed excitation loading p_0 = const. [Pa] on circle surface πc^2 with center in the point x_F, y_F is the coefficient p_{mn}

$$p_{mn} = \int_0^a \int_0^b P(x;y) X(\alpha x) Y(\beta_n y) dx\, dy$$

given by expression

$$p_{mn} = \frac{2F_0}{\gamma_{mn}} J_1(\gamma_{mn} c) \sin(\alpha_m x_F) \sin(\beta_n y_F) \tag{12}$$

where $\gamma_{mn} = \sqrt{\alpha_m^2 + \beta_n^2}$ pro $F_0 = p_0 \pi . c^2$

and $J_1(\gamma_{mn} c)$ is Bessel's first rank function, first order for argument $\gamma_{mn} c$

and next

$$\int_0^a \int_0^b \left[X(\alpha_m x) Y(\beta_n y) \right]^2 dx\, dy = \frac{a\, b}{4}$$

in this case it is possible to write coefficient A_5 in form

$$A_5 = \frac{8F_0}{a\, b \rho\, h\, c} J_1(\gamma_{mn} c) \sin(\alpha_m x_F) \sin(\beta_n y) \tag{12a}$$

It is suitable to solute the integral-differential equation (10) by application of the Laplace's transformation. After transformation we arise to

$$s^2\, \overline{W}(s) + A_1 \overline{W}(s) - A_2 \frac{\overline{W}(s)}{s + \delta_x} - A_3 \frac{\overline{W}(s)}{s + \delta_y} - A_4 \frac{\overline{W}(s)}{s + \delta_{xy}} = A_5 - \overline{T}_F(s)$$

which could be arranged in to the form

$$\overline{W}(s) = A_5 \overline{T}_F(s) F(s)$$

where $$F(s) = \frac{s^3 + a_2 s^2 + a_1 s + a_0}{\sum_{i=0}^{5} b_{5-i} . s^{5-i}} \tag{13}$$

where $a_2 = \delta_x + \delta_y + \delta_{xy}$, $a_1 = \delta_x \delta_y + \delta_x \delta_{xy} + \delta_y \delta_{xy}$, $a_0 = \delta_x \delta_y \delta_{xy}$

and next $b_5 = 1$, $b_4 = a_2$, $b_2 = a_0 + a_2 A_1 - A_2 - A_3 - A_4$,

$$b_3 = a_1 + A_1, \quad b_1 = a_1 A_1 - A_2(\delta_y + \delta_{xy}) - A_3(\delta_x + \delta_{xy}) - A_4(\delta_x + \delta_y) \tag{13a}$$

$$b_0 = a_0 A_1 - A_2 \delta_y \delta_{xy} - A_3 \delta_x \delta_{xy} - A_4 \delta_x \delta_y$$

For the reversed transformation it is suitable to arrange the function $\overline{F}(s)$ by method of undefined coefficient to the form of partial fraction $\overline{F}(s) = \sum_{i=1}^{n} \overline{F}_i(s)$.

Therefore it is necessary to determined roots of polynomial of the fraction denominator (13). It is possible to assume the three variation of the equation roots $\sum_{i=0}^{5} b_{5-i}.s^{5-i} = 0$

1) two complex conjugated roots

$$s_{1,2} = \beta_1 \pm i\,\omega_1\,, \qquad s_{3,4} = \beta_2 \pm i\,\omega_2\,,$$

where $\beta_1 = |\mathrm{Re}\, s_1 \langle 0|$, $\omega_1 = \mathrm{Im}\, s_1 \rangle 0$, $\beta_2 = |\mathrm{Re}\, s_2 \langle 0|$, $\omega_2 = \mathrm{Im}\, s_2 \rangle 0$

and one real $s_5 \langle 0 \rightarrow \beta_3 = |s_5 \langle 0|$

2) one zero complex conjugated point

$$s_{1,2} = \beta_1 \pm i\,\omega_1\,, \qquad \beta_1 = |\mathrm{Re}\, s_1 \langle 0|, \qquad \omega_1 = \mathrm{Im}\, s_1 \rangle 0$$

and three real roots $\beta_3 = |s_3 \langle 0|$, $\beta_4 = |s_4 \langle 0|$, $\beta_5 = |s_5 \langle 0|$

3) five real roots s_1, s_2, s_3, s_4, s_5 for $s_i \langle 0$

then $\beta_1 = |s_1|$, $\beta_2 = |s_2|$, $\beta_3 = |s_3|$, $\beta_4 = |s_4|$, $\beta_5 = |s_5|$

For these three cases it is necessary to perform reversed.

To 1) In first case it is possible to express the function $\overline{F}(s)$ in form

$$\overline{F}(s) = \overline{F}_1(s) + \overline{F}_2(s) + \overline{F}_3(s)$$

then

$$\frac{s^3 + a_2 s^2 + a_1 s + a_0}{\sum_{i=0}^{5} b_{5-i}.s^{5-i}} = \frac{C_1 s + D_1}{s^2 + p_1 s + q_1} + \frac{C_2 s + D_1}{s^2 + p_2 s + q_2} + \frac{C_3}{\rho - \beta_3}$$

where $p_i = -2\beta_i$, $\beta_i = |\mathrm{Re}\, s_i \langle 0|$, $q_i = \Omega_1^2 = \omega_i^2 + \beta_i^2$, $\omega_i = \mathrm{Im}\, s_i \rangle 0$ pro i= 1,2

Coefficitents C_1, C_2, C_3, D_1, D_2 are determined from linear equations system

$$\begin{aligned}
& C_1 + C_2 + C_3 = 0 \\
& -C_1^2 \beta_2 - C_2^2 \beta_1 - C_3 (2\beta_1 + 2\beta_2) + D_1 + D_2 = 1 \\
& C_1 \Omega_{0,2}^2 + C_2 \Omega_{01}^2 + C_3 (\Omega_{01}^2 + \Omega_{02}^2 + 2\beta_1\, 2\beta_2) - D_1 \beta_3 - D_2 \beta_3 = a_2 \\
& -C_3 (2\beta_1 \Omega_{02}^2 + 2\beta_2 \Omega_{01}^2) + D_1 2\beta_1 \beta_3 + D_2 2\beta_1 \beta_3 = a_1 \\
& C_3 \Omega_{01}^2 \Omega_{02}^2 + D_1 \Omega_{02}^2 \beta_3 + D_2 \Omega_{01}^2 \beta_3 = a_0
\end{aligned} \tag{14}$$

After reversed transformation of the function $\overline{W}(s)$ the expression is derived

$$W(t) = A_5 \int_0^t T_F(\tau) \left[\sum_{i=1}^{2} e^{-\beta_i (t-\tau)} \left(C_i \cos \omega_i (t-\tau) + \frac{D_i - C_i \beta_i}{\omega_i} \sin \omega_i (t-\tau) \right) + C_3 e^{-\beta_3 (t-\tau)} \right] d\tau \tag{15}$$

To 2) in the second case it is possible to express function $\overline{F}(s)$ in the form

$$\overline{F}(s)=\frac{C_1 s+D_1}{s^2+p_1 s+q_1}+\frac{C_2}{s-\beta_2}+\frac{C_3}{s-\beta_3}+\frac{C_4}{s-\beta_4}$$

where $p_1=-2\beta_1$, $\beta_1=\left|\operatorname{Re} s_1\langle 0\right|$, $\omega_1=\operatorname{Im} s_1\rangle 0$ $q_1=\Omega_{01}^2=\omega_1^2+\beta_1^2$

$\beta_2=\left|s_3\langle 0\right|$, $\beta_3=\left|s_4\langle 0\right|$, $\beta_4=\left|s_5\langle 0\right|$

Coefficients C_1, C_2, C_3, D_1, D_2 are determined from linear equations system

$$C_1+C_2+C_3+C_4=0$$

$$-C_1(\beta_1+\beta_2+\beta_3)-C_2(2\beta_1+\beta_3+\beta_4)-C_3(2\beta_1+\beta_2+\beta_4)-C_4(2\beta_1+\beta_2+\beta_3)+D_1=1$$

$$C_1(s_2\beta_3+\beta_2\beta_4+\beta_3\beta_4)+C_2(\Omega_{01}^2+\beta_3\beta_4+2\beta_1(\beta_3+\beta_4))+C_3(\Omega_{01}^2+2\beta_1(\beta_2+\beta_4)+\beta_2\beta_4)+$$

$$+C_4(\Omega_{01}^2+2\beta_1(\beta_2+\beta_3)+\beta_2\beta_3)-D_1(\beta_2+\beta_3+\beta_4)=a_2$$

$$-C_1\beta_2\beta_3\beta_4-C_2\left[2\beta_1\beta_3\beta_4+\Omega_{01}^2(\beta_3+\beta_4)\right]-C_3\left[2\beta_1\beta_3\beta_4+\Omega_{01}^2(\beta_2+\beta_4)\right]-$$

$$-C_4\left[2\beta_1\beta_2\beta_3+\Omega_{01}^2(\beta_2+\beta_3)\right]=a_1 \tag{16}$$

$$C_2\Omega_{01}^2\beta_3\beta_4+C_3\Omega_{01}^2\beta_2\beta_4+C_4\Omega_{01}^2\beta_2\beta_3-D_1\beta_2\beta_3\beta_4=a_0$$

After reversed transformation of the function $\overline{W}(s)$ the expression is derived

$$W(t)=A_5\int_0^t T_F(\tau)\left[e^{-\beta_1(t-\tau)}\left(C_1\cos\omega_1(t-\tau)+\frac{D_1-C_1\beta_1}{\omega_1}\sin\omega_1(t-\tau)\right)+\sum_{i=2}^{4}c_i e^{-\beta_1(t-\tau)}\right]d\tau \tag{17}$$

To 3) In third case it is possible to express the function $\overline{F}(s)$ in form

$\overline{F}(s)=\sum_{i=1}^{5}\frac{c_i}{s-\beta_i}$, where $\beta_i=\left|s_i\langle 0\right|$, pro $i=1,2,3,4,5$

coefficients $C_i i=1,\dots,5$

$$\sum_{i=1}^{5}C_i=0,\quad \sum_{i=1}^{5}C_i\sum_{\substack{j=1\\ j\neq i}}^{5}\beta_j=1,\quad \sum_{i=1}^{5}C_i\sum_{\substack{j=1\\ j\neq i}}^{5}\beta_j\sum_{\substack{l=j+1\\ l\neq i}}^{4}\beta_l=a_2,\quad \sum_{i=1}^{5}C_i\sum_{\substack{j=1\\ j\neq i}}^{5}\beta_j\sum_{\substack{l=j+1\\ l\neq i}}^{4}\beta_l\sum_{\substack{k=l+1\\ k\neq i}}^{5}\beta_k=a_1$$

$$\sum_{i=1}^{5}C_i\prod_{\substack{j=1\\ j\neq i}}^{5}\quad=a_0 \tag{18}$$

After reversed transformation of the function $\overline{W}(s)$ the expression is derived

$$W(t)=A_5\int_0^t\sum_{i=1}^{5}C_i.e^{-\beta_i(t-\tau)}d\tau \tag{19}$$

The demanded function *W(t)* is possible to express in followed form. The function is depended on the input values (material and geometric), it means the coefficient bi (13a) of the polynomial of the fraction denominator (13). This coefficient determine values of the zero point s and polynomial's points.

$$W(t) = A_5 \int_0^t T_F(\tau) K(t-\tau)\, d\tau \tag{20}$$

where function *K(t)* is in particular cases for $\beta_i = |\mathrm{Re}\, s_i \langle 0|$, $\omega_i = \mathrm{Im}\, s_i \rangle 0$

1) $$K_1(t) = \sum_{i=1}^{2} e^{-\beta_i t}\left(C_i \cos\omega_i t + \frac{D_i - C_i\beta_i}{\omega_i}\sin\omega_i t \right) + C_3.e^{-\beta_3 t}, \tag{21a}$$

where C_i, D_i see (14)

2) $$K_2(t) = e^{-\beta_1 t}\left(C_1 \cos\omega_1 t + \frac{D_1 - C_1\beta_1}{\omega_1}\sin\omega_1 t \right) + \sum_{i=2}^{4} C_i.e^{-\beta_i t}, \tag{21b}$$

where C_1, D_1, C_i for $i = 2,3,4$ see (16)

3) $$K_3(t) = \sum_{i=2}^{5} C_i.e^{-\beta_i t}, \qquad \text{i = 1,2,3,4,5,} \tag{21c}$$

where C_i see (18)

The demanded vertical displacement function *w(x;y;t)* according to (10) is given by followed expression in solved case

$$w(x;y;t) = \frac{8F_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\sin(\alpha_m x).$$

$$.\sin(\beta_n y)\int_0^t T_F(t) K(t-\tau)\, d\tau, \qquad \text{pro } i = 1, 2, 3 \tag{22}$$

If the time course of the external loading *F*(*t*) defined by Heavisid's function, where unit jump $T_F(t) = H(t)$, then time function is

$$T_K(t) = \int_0^t H(\tau)\, K(t-\tau)\, d\tau$$

and its derivation is $\frac{\partial}{\partial t}T_k(t)$ for determination of the velocity components of the displacement for particular cases, the function *K(t)* is expressed by equations

To 1) for $K_1(t)$:

$$T_{K1}(t) = \sum_{i=1}^{2}\frac{\omega_i}{\omega_i^2 + \beta_i^2}\left\{ \frac{D_i}{\omega_i}\left(1 - e^{-\beta_1 t}\cos\omega_i t\right) + \left[C_i\left(\left(\frac{\beta_i}{\omega_i}\right)^2 + 1\right) - \frac{D_i}{\omega_i}.\frac{\beta_i}{\omega_i} \right] e^{-\beta_i t}\sin\omega_i t \right\} +$$

$$+\frac{C_3}{\beta_3}\left(1 - e^{-\beta_3 t}\right)$$

$$\frac{\partial T_{K1}(t)}{\partial t}=\sum_{i=1}^{2}e^{-\beta_i t}\left[C_i\cos\omega_i t+\left(\frac{D_i}{\omega_i}-C_i\frac{\beta_i}{\omega_i}\right)\sin\omega_i t\right]+C_3e^{-\beta_3 t} \tag{23a}$$

To 2) for *K₂(t):*

$$T_{K2}(t)=\frac{\omega_1}{\omega_1^2+\beta_1^2}\left\{\frac{D_i}{\omega_i}\left(1-e^{-\beta_1 t}\cos\omega_1 t\right)+\left[C_1\left(\left(\frac{\beta_1}{\omega_1}\right)^2+1\right)-\frac{D_1}{\omega_1}.\frac{\beta_1}{\omega_1}\right]e^{-\beta_i t}\sin\omega_1 t+\sum_{i=2}^{4}C_ie^{-\beta_i t}\right. \text{,(23b)}$$

$$\frac{\partial T_{K2}(t)}{\partial t}=e^{-\beta_1 t}\left[C_1\cos\omega_1 t+\left(\frac{D_1}{\omega_1}-\frac{\beta_1}{\omega_1}\right)\sin\omega_1 t\right]+\sum_{i=2}^{4}C_ie^{-\beta_i t}$$

To 3) for *K₃(t):*

$$T_{K3}(t)=\sum_{i=1}^{5}\frac{C_i}{\beta_i}\left[1-e^{-\beta_i t}\right] \tag{23c}$$

$$\frac{\partial T_{K3}(t)}{\partial t}=\sum_{i=1}^{5}C_ie^{-\beta_i t}$$

Resulting equation for vertical displacement *w(x;y;t)* solution for given rectangular orthotropic 2D plate for Kirchhoff's model, Rayleygh's model of plate, viscoelastic plate for Zener's model of standard body, simply supported, loaded on the circle surface $\pi.c^2$ by continual loading with time dependency corresponding to Heavisid's jump function, could be expressed in form

$$w(x;y;t)=\frac{8F_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\,\sin(\alpha_m x)\,\sin(\beta_n y)T_K(t) \tag{24}$$

velocity of the vertical displacement

$$\dot{w}(x;y;t)=\frac{8F_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\sin(\alpha_m x)\,\sin(\beta_n y)\frac{\partial T_k(t)}{\partial t}$$

Equations for solution of the horizontal displacements component

$$u(x;y;z;t)=-\frac{z\partial w}{\partial x}=-\frac{8zF_o}{a.b.\rho.h.c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\cos(\alpha_m x).$$

$$.\sin(\beta_n y)T_k(t)$$

$$v(x;y;z;t)=-\frac{z\partial w}{\partial y}=-\frac{8zF_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta_n J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\sin(\alpha_m x).$$

$$.\cos(\beta_n y)T_k(t)$$

Equations for solution of the velocity components of the horizontal displacements

$$\dot{u}(x;y;z;t)=-\frac{z\partial^2 w}{\partial x\partial t}=-\frac{8zF_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\cos(\alpha_m x).$$

$$.\sin(\beta_n y)\frac{\partial T_k(t)}{\partial t}$$

$$\dot{v}(x;y;z;t) = -\frac{z\partial^2 w}{\partial y \partial t} = -\frac{8zF_o}{a\,b\,\rho\,h\,c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta_n J_1(\gamma_{mn}c)}{\gamma_{mn}\psi_{mn}}\sin(\alpha_m x_F)\sin(\beta_n y_F)\sin(\alpha_m x).$$

$$.\cos(\beta_n y)\frac{\partial T_k(t)}{\partial t}$$

Where function $T_K(t)$ and its derivation $\frac{\partial T_K(t)}{\partial t}$ are given for particular cases K_1, K_2, K_3 by expression (23)

3. Conclusion

Equations for solution of stress components are determined substitution of equation (22), or for $T_F(t) = H(t)$ and equation (24) into the expression (5).

Evaluation derived expression and numerical solution mentioned problem by FEM in system MSC and Matlab is provided on UT AV ČR in Plzeň. The solution results both of these methods, it means approximated analytical and FEM, included theirs comparison, will be proposed during contribution presentation at conference.

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