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ON APPLICATIONS OF GENERALIZED FUNCTIONS TO CALCULATION OF THIN-WALLED BEAM DESIGN ELEMENTS SUBJECT TO NON-UNIFORM TORSION

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Summary: *Vlasov's mathematical model of the restrained torsion of a prismatic thin-walled open-section beam contains derivatives of unknown functions which are the torque, the bimoment, the twist, and the rotation of cross sections. But these derivatives are not defined at such points between ends of a beam where a concentrated torsional load or a concentrated bimoment load or an internal support or an internal coupling is located. In order that the Vlasov's mathematical model of thin-walled open-section elastic beam subjected to torsion allowing for constrained warping may hold true also at the points of discontinuity mentioned, which are common in calculating experience, we have used the distributional derivatives for the unknown quantities, and developed generalized mathematical model in the form of a system of ordinary differential equations (SODE). In order to solve this generalized model we use the Laplace transform, and find the general solution which is the generalization of the initial parameters' method since it covers also bars with internal couplings.*

1. Introduction

The beams with discontinuities in loading or geometry are usually calculated in such a way that they are at first divided into subintervals without the discontinuities. Then continuous solutions with integration constants are determined for every such subinterval apart. Finally the integration constants are determined from boundary and continuity conditions.

In this paper we use Dirac distribution and the Heaviside's unit step function in order to derive a generalized Vlasov's model of the restrained torsion of a prismatic thin-walled open-section beam in the form of a system of ordinary differential equations (SODE) (19) to (22) which covers the discontinuities in loading and geometry. This generalized mathematical model of restrained torsion may be solved like only one differential problem without dividing the beam into subintervals and without continuity conditions.

A discontinuity in the twist may occur at a point in which a rigid coupling between bar segments is located. Discontinuities in the rotation of cross sections and in the twist may be found at a point in which an elastic coupling between bar segments is situated. Magnitudes of

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these discontinuities can be determined through the use of deformation conditions. There is zero axial stress acting on the face of bar segments at a coupling. Hence the bimoment is zero at the coupling.

Discontinuities in distributed torsional load m(x) may be expressed by means of Heaviside's unit step function moved to points of the discontinuities.

2. The scope of the validity of Vlasov's mathematical model of the restrained torsion of a prismatic thin-walled open-section beam

Vlasov's mathematical model of the restrained torsion of a prismatic thin-walled open-section beam may be expressed in the form of the first order ordinary differential equation system as follows

$$\frac{d}{dx} M_T(x) = -\mathrm{m}(x) \tag{1}$$

$$\frac{d}{dx} \mathrm{B}(x) = M_T(x) - G\, J_T\, \theta(x) \tag{2}$$

$$\frac{d}{dx} \theta(x) = -\frac{\mathrm{B}(x)}{E\, J_\omega} \tag{3}$$

$$\frac{d}{dx} \phi(x) = \theta(x) \tag{4}$$

where

$M_T(x)$	the torque [*Nm*],
B(x)	the bimoment [Nm^2],
$\theta(x)$	the twist [$radm^{-1}$],
$\phi(x)$	the rotation of cross sections [rad] ,
m(x)	a distributed torsion load [*N*],
J_T	polar moment of inertia [m^4],
J_ω	warping resistance [m^6],
E	Young's modulus of elasticity [*Pa*],
G	shear modulus of elasticity [*Pa*].

The SODE (1) to (4) is valid in the space of real functions providing that all the derivatives exist. However the unknown functions $M_T(x)$, $\mathrm{B}(x)$, $\theta(x)$, $\phi(x)$ may have discontinuities which are common in calculating experience. The SODE (1) to (4) does not hold good at such points between ends of a bar in which the unknown functions have discontinuities.

When we need to express a derivative of a discontinuous quantity we must use a space of distributions with a distributional derivative instead of the space of real functions. The distributional derivative of a discontinuous generalized function f(x) which has one jump discontinuity at $x = x_0$ is expressed as follows:

$$f' = \{f'(x)\} + [f]_{x_0} \,.\, \mathrm{Dirac}(x - x_0) \tag{5}$$

where

$\{f'(x)\}$	classical derivative,
$[f]_{x_0}$ = f(x_0 + 0) - f(x_0 - 0)	magnitude of the jump discontinuity of f(x) at $x = x_0$,
$\mathrm{Dirac}(x - x_0)$	Dirac distribution moved to $x = x_0$.

3. The distributional derivative of the unknown quantities with one jump discontinuity

Let the torque $M_T(x)$ have a jump discontinuity at point x = a_1 of magnitude:

$$M_T(a_1 + 0) - M_T(a_1 - 0) = -M_1 \qquad (6)$$

as a result of a concentrated force couple action. The equation (6) stands for equilibrium of an infinitesimal element which was cut out with the concentrated force couple. Then the distributional derivative (5) of $M_T(x)$ is

$$M_T' = -\mathrm{m}(x) - M_1\, \mathrm{Dirac}(x - a_1)\ . \qquad (7)$$

Let the bimoment B(x) have a jump discontinuity at $x = b_1$ of magnitude:

$$\mathrm{B}(b_1 + 0) - \mathrm{B}(b_1 - 0) = -B_1\ . \qquad (8)$$

as a result of a line force $p_1(s)$ acting in the axial direction of a thin-walled beam along the middle line at $x = b_1$. The equation (8) was obtained from force equilibrium of an infinitesimal element that was cut out from middle surface of width ds for dx = 0:

$$\sigma_x(x = b_1 + 0, s)\, dA + p_1(s)\, ds - \sigma_x(x = b_1 - 0, s)\, dA = 0 \quad , \qquad (9)$$

which may be multiplied by a warping function ω(s), and integrated along the cross-sectional centerline from s = 0 to s = n as follows:

$$\int_0^n \sigma_x(x = b_1 + 0, s)\, \omega(s)\, \delta(s)\, ds + \int_0^n p_1(s)\, \omega(s)\, ds - \int_0^n \sigma_x(x = b_1 - 0, s)\, \omega(s)\, \delta(s)\, ds = 0 \quad , \qquad (10)$$

where

dA	differential of the cross-sectional area [m^2],
$\sigma_x(x, s)$	axial stresses due to restrained warping [Pa],
$p_1(s)$	axial line force acting along the cross-sectional centerline at $x = b_1$ [Nm^{-1}],
ω(s)	warping function [m^2] ,
$\delta(s)$	thickness of the profile [m],
s	curvilinear coordinate along the middle line (cross-sectional centerline) [m].

Thus the distributional derivative of the bimoment is as follows

$$\mathrm{B}' = M_T(x) - G\, J_T\, \theta(x) - B_1 \,\mathrm{Dirac}(x - b_1)\,, \tag{11}$$

where the concentrated bimoment load is determined as

$$B_1 = \int_0^n p_1(s)\, \omega(s)\, ds \quad . \tag{12}$$

Let the twist have a jump discontinuity at $x = c_1$ of magnitude

$$\theta(c_1 + 0) - \theta(c_1 - 0) = \Theta_1 \tag{13}$$

owing to a coupling placed between ends of a beam. Thus the distributional derivative of twist is as follows:

$$\theta\,' = -\frac{\mathrm{B}(x)}{E\, J_\omega} + \Theta_1 \,\mathrm{Dirac}(x - c_1) \quad . \tag{14}$$

The unknown magnitude Θ_1 is determined from a condition that the internal coupling does not transmit the internal bimoment because there are zero axial stresses on the coupling faces at $x = c_1$.

Let the rotation of cross sections have a jump discontinuity at $x = d_1$ of magnitude:

$$\phi(d_1 + 0) - \phi(d_1 - 0) = \Phi_1 \tag{15}$$

by virtue of an elastic coupling with a torsional stiffness k_T situated between ends of a beam. Hence the distributional derivative of the rotation of cross sections is as follows

$$\phi\,' = \theta(\mathrm{x}) + \Phi_1 \,\mathrm{Dirac}(x - d_1) \quad . \tag{16}$$

The unknown magnitude Φ_1 is determined from a condition of the elastic coupling:

$$M_T(d_1 + 0) = k_T\, \Phi_1 \quad . \tag{17}$$

This condition holds good also in such a case that there is a concentrated force couple connected to the internal coupling from the left. If the concentrated force couple is connected to the right part of the internal coupling, then a modified deformation condition has to be used as follows:

$$M_T(d_1 - 0) = k_T\, \Phi_1 \quad . \tag{18}$$

4. Generalized SODE of Vlasov's mathematical model of the restrained torsion of a prismatic thin-walled open-section beam

Let the torque $M_T(x)$ have jump discontinuities: M_1 at $x = a_1$, M_2 at $x = a_2$, …, M_{n_1}

at $x = a_{n_1}$, where $0 < a_1$, $0 < a_2$, ..., $0 < a_{n_1}$.

Let the bimoment B(x) have jump discontinuities: B_1 at $x = b_1$, B_2 at $x = b_2$, …, B_{n_2} at $x = b_{n_2}$, where $0 < b_1$, $0 < b_2$, … , $0 < b_{n_2}$.

Let the twist $\theta(x)$ have jump discontinuities: Θ_1 at $x = c_1$, Θ_2 at $x = c_2$, …, Θ_{n_3} at $x = c_{n_3}$, where $0 < c_1$, $0 < c_2$, … , $0 < c_{n_3}$.

Let the rotation of cross sections $\phi(x)$ have jump discontinuities: Φ_1 at $x = d_1$, Φ_2 at $x = d_2$, …, Φ_{n_4} at $x = d_{n_4}$, where $0 < d_1$, $0 < d_2$, … , $0 < d_{n_4}$.

If we now generalize the distributional derivatives (7), (11), (14), (16) for final number of jump discontinuities, then we obtain a generalized SODE of Vlasov's model of the restrained torsion of a thin-walled beam as follows

$$\frac{d}{dx} M_T(x) = -\mathrm{m}(x) - \left(\sum_{i=1}^{n_1} M_i \,\mathrm{Dirac}(x - a_i) \right) \tag{19}$$

$$\frac{d}{dx} \mathrm{B}(x) = M_T(x) - G\, J_T\, \theta(x) - \left(\sum_{i=1}^{n_2} B_i \,\mathrm{Dirac}(x - b_i) \right) \tag{20}$$

$$\frac{d}{dx} \theta(x) = -\frac{\mathrm{B}(x)}{E\, J_\omega} + \left(\sum_{i=1}^{n_3} \Theta_i \,\mathrm{Dirac}(x - c_i) \right) \tag{21}$$

$$\frac{d}{dx} \phi(x) = \theta(x) + \left(\sum_{i=1}^{n_4} \Phi_i \,\mathrm{Dirac}(x - d_i) \right) \tag{22}$$

5. Solution procedure of the generalized SODE of Vlasov's model of the restrained torsion of a thin-walled open-section prismatic beam

The general solution to the generalized SODE (19), (20), (21), (22) is determined by means of the Laplace transform in three steps:

a) Transform differential equations (19), (20), (21), (22) into an algebraic system.

b) Find a solution to the algebraic system.

c) Apply the inverse Laplace transformation to the solution of the algebraic system.

5.1. The Laplace transform of the SODE (19), (20), (21), (22) [$\lambda^2 = G\, J_T / (E\, J_\omega)$]

$$p\,\mathrm{laplace}(M_T(x), x, p) - M_T(0) = -\mathrm{laplace}(\mathrm{m}(x), x, p) - \left(\sum_{i=1}^{n_1} M_i\, \mathbf{e}^{(-p\, a_i)} \right) \tag{23}$$

$$p\,\mathrm{laplace}(\mathrm{B}(x),x,p)-\mathrm{B}(0)=$$
$$\mathrm{laplace}(M_T(x),x,p)-\lambda^2\,E\,J_\omega\,\mathrm{laplace}(\theta(x),x,p)-\left(\sum_{i=1}^{n_2} B_i\,\mathbf{e}^{(-p\,b_i)}\right) \tag{24}$$

$$p\,\mathrm{laplace}(\theta(x),x,p)-\theta(0)=-\frac{\mathrm{laplace}(\mathrm{B}(x),x,p)}{E\,J_\omega}+\left(\sum_{i=1}^{n_3}\Theta_i\,\mathbf{e}^{(-p\,c_i)}\right) \tag{25}$$

$$p\,\mathrm{laplace}(\phi(x),x,p)-\phi(0)=\mathrm{laplace}(\theta(x),x,p)+\left(\sum_{i=1}^{n_4}\Phi_i\,\mathbf{e}^{(-p\,d_i)}\right) \tag{26}$$

where

p	a variable for the Laplace transform
laplace	Laplace transform operator
laplace(f(x), *x*, *p*)	Laplace transform of f(x)
$M_T(0)$, B(0), $\theta(0)$, $\phi(0)$	constants of integration in the form of initial parameters

5.2 The Laplace transforms of unknown quantities

Solution to the algebraic system of equations (23), (24), (25), (26) is:

$$\mathrm{laplace}(M_T(x),x,p)=\frac{M_T(0)}{p}-\frac{\mathrm{laplace}(\mathrm{m}(x),x,p)}{p}-\left(\sum_{i=1}^{n_1}\frac{M_i\,\mathbf{e}^{(-p\,a_i)}}{p}\right) \tag{27}$$

$$\mathrm{laplace}(\mathrm{B}(x),x,p)=\frac{M_T(0)}{p^2-\lambda^2}+\frac{p\,\mathrm{B}(0)}{p^2-\lambda^2}-\frac{\lambda^2\,E\,J_\omega\,\theta(0)}{p^2-\lambda^2}-\left(\sum_{i=1}^{n_1}\frac{\mathbf{e}^{(-p\,a_i)}\,M_i}{p^2-\lambda^2}\right)$$
$$-\left(\sum_{i=1}^{n_2}\frac{\mathbf{e}^{(-p\,b_i)}\,p\,B_i}{p^2-\lambda^2}\right)-\left(\sum_{i=1}^{n_3}\frac{\mathbf{e}^{(-p\,c_i)}\,E\,J_\omega\,\lambda^2\,\Theta_i}{p^2-\lambda^2}\right)-\frac{\mathrm{laplace}(\mathrm{m}(x),x,p)}{p^2-\lambda^2} \tag{28}$$

$$\mathrm{laplace}(\theta(x),x,p)=-\frac{M_T(0)}{E\,J_\omega\,(p^2-\lambda^2)\,p}-\frac{\mathrm{B}(0)}{E\,J_\omega\,(p^2-\lambda^2)}+\frac{p\,\theta(0)}{p^2-\lambda^2}$$
$$+\left(\sum_{i=1}^{n_1}\frac{\mathbf{e}^{(-p\,a_i)}\,M_i}{E\,J_\omega\,(p^2-\lambda^2)\,p}\right)+\left(\sum_{i=1}^{n_2}\frac{\mathbf{e}^{(-p\,b_i)}\,B_i}{E\,J_\omega\,(p^2-\lambda^2)}\right)+\left(\sum_{i=1}^{n_3}\frac{p\,\mathbf{e}^{(-p\,c_i)}\,\Theta_i}{p^2-\lambda^2}\right)$$
$$+\frac{\mathrm{laplace}(\mathrm{m}(x),x,p)}{p\,E\,J_\omega\,(p^2-\lambda^2)} \tag{29}$$

$$\text{laplace}(\phi(x), x, p) = -\frac{M_T(0)}{E\,J_\omega\,p^2\,(p^2-\lambda^2)} - \frac{\mathrm{B}(0)}{E\,J_\omega\,p\,(p^2-\lambda^2)} + \frac{\theta(0)}{p^2-\lambda^2} + \frac{\phi(0)}{p}$$
$$+\left(\sum_{i=1}^{n_1} \frac{\mathbf{e}^{(-p\,a_i)}\,M_i}{E\,J_\omega\,p^2\,(p^2-\lambda^2)}\right) + \left(\sum_{i=1}^{n_2} \frac{\mathbf{e}^{(-p\,b_i)}\,B_i}{E\,J_\omega\,p\,(p^2-\lambda^2)}\right) + \left(\sum_{i=1}^{n_3} \frac{\mathbf{e}^{(-p\,c_i)}\,\Theta_i}{p^2-\lambda^2}\right)$$
$$+\left(\sum_{i=1}^{n_4} \frac{\Phi_i\,\mathbf{e}^{(-p\,d_i)}}{p}\right) + \frac{\text{laplace}(\mathrm{m}(x), x, p)}{E\,J_\omega\,p^2\,(p^2-\lambda^2)} \tag{30}$$

5.3 The general solution to the generalized SODE [$\lambda^2 = G\,J_T/(E\,J_\omega)$]

The system of differential equations (19), (20), (21), (22) has the general solution (31), (32), (33), (34) which is determined like inverse Laplace transformation applied to images (27), (28), (29), (30).

$$M_T(x) = M_T(0) - \int_0^x \mathrm{m}(\xi)\,d\xi - \left(\sum_{i=1}^{n_1} M_i\,\text{Heaviside}(x-a_i)\right) \tag{31}$$

$$\mathrm{B}(x) = \frac{M_T(0)\sinh(\lambda\,x)}{\lambda} + \mathrm{B}(0)\cosh(\lambda\,x) - \lambda\,E\,J_\omega\sinh(\lambda\,x)\,\theta(0)$$
$$-\frac{\sum_{i=1}^{n_1} M_i\,\text{Heaviside}(x-a_i)\sinh(\lambda\,(x-a_i))}{\lambda}$$
$$-\left(\sum_{i=1}^{n_2} B_i\,\text{Heaviside}(x-b_i)\cosh(\lambda\,(x-b_i))\right)$$
$$-\lambda\,E\,J_\omega\left(\sum_{i=1}^{n_3} \text{Heaviside}(x-c_i)\,\Theta_i\sinh(\lambda\,(x-c_i))\right)$$
$$+\frac{\frac{1}{2}\int_0^x \mathrm{m}(\xi)\,\mathbf{e}^{(\lambda\,(-x+\xi))}\,d\xi - \left(\frac{1}{2}\int_0^x \mathrm{m}(\xi)\,\mathbf{e}^{(-\lambda\,(-x+\xi))}\,d\xi\right)}{\lambda} \tag{32}$$

$$
\begin{aligned}
\theta(x) = {} & \frac{(1-\cosh(\lambda x))\, M_T(0)}{J_\omega E \lambda^2} - \frac{\sinh(\lambda x)\, \mathrm{B}(0)}{E J_\omega \lambda} + \theta(0)\cosh(\lambda x) \\
& + \frac{\sum_{i=1}^{n_1} 2\sinh\left(\frac{\lambda (x-a_i)}{2}\right)^2 \mathrm{Heaviside}(x-a_i)\, M_i}{J_\omega E \lambda^2} \\
& + \frac{\sum_{i=1}^{n_2} \mathrm{Heaviside}(x-b_i)\sinh(\lambda (x-b_i))\, B_i}{E J_\omega \lambda} \\
& + \left(\sum_{i=1}^{n_3} \Theta_i\, \mathrm{Heaviside}(x-c_i)\cosh(\lambda (x-c_i))\right) \\
& + \frac{\frac{1}{2}\int_0^x \mathrm{m}(\xi)\, \mathbf{e}^{(-\lambda(-x+\xi))}\, d\xi + \frac{1}{2}\int_0^x \mathrm{m}(\xi)\, \mathbf{e}^{(\lambda(-x+\xi))}\, d\xi - \int_0^x \mathrm{m}(\xi)\, d\xi}{J_\omega E \lambda^2}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\phi(x) = {} & \left(\frac{x}{J_\omega E \lambda^2} - \frac{\sinh(\lambda x)}{J_\omega E \lambda^3}\right) M_T(0) + \frac{(1-\cosh(\lambda x))\, \mathrm{B}(0)}{J_\omega E \lambda^2} + \frac{\sinh(\lambda x)\, \theta(0)}{\lambda} + \phi(0) \\
& - \left(\sum_{i=1}^{n_1} \left(\frac{x-a_i}{J_\omega E \lambda^2} - \frac{\sinh(\lambda (x-a_i))}{J_\omega E \lambda^3}\right) \mathrm{Heaviside}(x-a_i)\, M_i\right) \\
& + \left(\sum_{i=1}^{n_2} \frac{2\sinh\left(\frac{\lambda (x-b_i)}{2}\right)^2 \mathrm{Heaviside}(x-b_i)\, B_i}{J_\omega E \lambda^2}\right) \\
& - \frac{\sum_{i=1}^{n_3} \mathrm{Heaviside}(x-c_i)\, \Theta_i \sinh(\lambda (-x+c_i))}{\lambda} + \left(\sum_{i=1}^{n_4} \mathrm{Heaviside}(x-d_i)\, \Phi_i\right) \\
& + \frac{1}{E J_\omega \lambda^2}\int_0^x \mathrm{m}(\xi)(-x+\xi)\, d\xi + \frac{\int_0^x \mathrm{m}(\xi)\, \mathbf{e}^{(-\lambda(-x+\xi))}\, d\xi - \int_0^x \mathrm{m}(\xi)\, \mathbf{e}^{(\lambda(-x+\xi))}\, d\xi}{2 E J_\omega \lambda^3}
\end{aligned}
\tag{34}
$$

6. Conclusions

The contribution of this paper is that the generalized SODE (19) to (22) holds true also for discontinuous unknown quantities, and that its general solution (31) to (34) stands good for bars containing internal rigid or elastic couplings as well. The general solution (31) to (34) was found by means of the Laplace transform and with using the symbolic programming approach. The integration constants are in the form of initial parameters. The general solution (31) to (34) is the generalization of the initial parameters method because it covers also bars with internal couplings.

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